Coisotropic Submanifolds and the BFV-Complex

Dissertation

zur
Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der
Mathematisch-naturwissenschaftlichen Fakultät
der
Universität Zürich

von

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aus
Österreich

Promotionskomitee

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Zürich, 2009

Synopsis

The object of primary interest of this thesis is a certain algebraic structure – known as the BFV-complex – associated to coisotropic submanifolds of Poisson manifolds. Coisotropic submanifolds form a natural class of subobjects of Poisson manifolds, e.g. Lagrangian submanifolds of symplectic manifolds, hypersurfaces, Lie subalgebras and graphs of Poisson maps yield examples of coisotropic submanifolds. Moreover, it was recently realized that coisotropic submanifolds arise in the study of physical objects known as "branes" in the framework of topological string theory ([**KO**]) and the Poisson sigma model ([**CF2**]). Any coisotropic submanifold comes along with a foliation and a certain quotient algebra which carries the structure of a Poisson algebra. There are two cochain complexes whose zero'th cohomologies are isomorphic to the quotient algebra: the Lie algebroid complex (see [**Mc**]) , and the BFV-complex (see [**BF**] and [**BV**]) respectively. Moreover it is well-known that the cohomologies of both complexes match in all degrees.

In [**OP**] and [**CF**] the Lie algebroid complex was enriched by higher order operations that encode the Poisson bracket on the quotient algebra. The higher order operations are not natural but depend on certain choices. However, we prove that different choices lead to isomorphic structures. On the other hand the BFV-complex carries the structure of a graded Poisson algebra which also encodes the Poisson bracket on the quotient algebra. The question arises how tightly the Lie algebroid complex enriched by higher order operations on the one hand and the algebraic structure on the BFV-complex on the other hand are related. We give an affirmative answer to this question by proving that these two structures are "isomorphic up to homotopy" – more technically speaking, they are L_∞ quasi-isomorphic.

Although the two structures described above are tightly related, there are some subtle difference between them. While the enriched Lie algebroid complex only captures information about the *jets* of the Poisson structure on the coisotropic submanifold under investigation, the BFV-complex yields an invariant that depends on the *germ* of the Poisson structure. This has drastic consequence for the associated smooth deformation problems: it is possible to understand the nearby deformations of the coisotropic submanifold with the help of the BFV-complex, while this is not possible in general if one uses the enriched Lie algebroid complex instead.

Let L be a Lagrangian submanifold of a symplectic manifold. As mentioned above, Lagrangian submanifolds are special cases of coisotropic submanifolds. It is well-known that the space of Lagrangian submanifolds near L is described by the de Rham complex of L. Moreover the moduli space of Lagrangian submanifolds near L corresponds to $H^1(L,\mathbb{R})$. One can try to generalize this in two ways. On the one hand, one can try to incorporate effects of a global nature. This is usually done by "counting" pseudo-holomorphic objects. This idea goes back to Gromov ([**Gr**]) and Floer ([**Fl**]) and was developed into a full-fledged theory in [**FOOO**].

On the other hand, one can try to understand the local deformation problem of more general objects than Lagrangian submanifolds. The deformation problem of coisotropic submanifolds of symplectic manifolds was considered in [**OP**] and a description of nearby coisotropic submanifolds in terms of the enriched Lie algebroid complex was given. However, this connection between the deformation problem of coisotropic submanifolds and the enriched Lie algebroid complex fails in the category of Poisson manifolds.

We give a description of the set of nearby coisotropic submanifolds with the help of the BFV-complex, which holds even for coisotropic submanifolds of Poisson manifolds. Moreover a "categorification" of the moduli space of nearby coisotropic submanifolds is introduced. This yields a certain groupoid. A second groupoid is constructed using data of the BFV-complex only. Moreover this groupoid comes along with a natural surjective morphism to the groupoid of nearby coisotropic submanifolds. The kernel of this morphism can be described in terms of the BFV-complex and as a corollary we obtain a bijection between the moduli space of nearby coisotropic sections and the moduli space of certain Maurer–Cartan elements of the BFV-complex.

The connection between the two groupoids mentioned above can be read in two directions. First it provides a new approach to the deformation problem of coisotropic submanifolds. Second it clarifies the geometric information encoded in the BFV-complex which is usually introduced as the starting point for a very different problem: the (geometric or deformation) quantization of coisotropic submanifolds.

For a more detailed exposition of the content of this thesis, we refer the reader to the Introduction, see Chapter 1.

Parts of the material presented here appeared in [**Sch1**], [**Sch2**] and [**Sch3**] and in the joint article [**CS**] with Cattaneo respectively.

The structure of this thesis is a follows: In Chapter 1 we give a introduction to the thesis. We briefly explain the most relevant concepts – Poisson manifolds, Hamiltonian diffeomorphisms, and coisotropic submanifolds – and state the main motivating problem: to find a description of the set of coisotropic submanifolds near a given one. In examples we demonstrate the connection to the Lie algebroid complex and the homotopy Lie algebroid. Both structures contain important information but are not sufficient to solve the main problem. Finally we give a rough description of the BFV-complex and explain how one can solve the main problem with its help.

In Chapter 2 we introduce the necessary background material. In particular we recall the basic definitions concerning L_∞-algebras (Section 1), explain the transfer of differential graded Lie algebras along contractions (Section 2), give an account of the higher derived brackets formalism (Section 3) and explain parts of the theory of smooth graded manifolds (Section 4).

Chapter 3 covers basic facts about Poisson geometry (Section 1), introduces coisotropic submanifolds (Section 2) and explains the construction of the enriched Lie algebroid complex (Section 3).

In Chapter 4 we introduce the BFV-complex and establish some of its properties. As a preparation we construct a certain L_∞ quasi-isomorphism from the space of multivector fields on a manifold M to the space of multivector fields on a smooth graded manifold with body M. In Section 2 we use this L_∞ quasi-isomorphism to give a conceptual construction of the BFV-bracket. Moreover we recall the construction of a differential on the BFV-complex. This differential and the BFV-bracket equip the BFV-complex with the structure of a differential graded Poisson algebra. The dependence of this algebraic structure on the choices involved in its construction is clarified. In Section 3 the connection between the BFV-complex on the one hand and the enriched Lie algebroid complex on the other hand is established, see Theorem 3.6.

The geometric content of the BFV-complex is investigated in more detail in Chapter 5. The BFV-complex comes along with a certain equation – called the Maurer–Cartan equation – and we study the set of solutions of this equation in Section 1. It turns out that this set is tightly related to the set of coisotropic sections, see Theorem 1.13. In Section 2 we take the inner symmetries of the Poisson manifold and of the differential graded Poisson algebra structure on the BFV-complex into account. We conclude with Theorem 2.25 where the moduli space of coisotropic sections is identified with the moduli space of certain Maurer–Cartan elements of the BFV-complex. In Section 3 this relation is "categorified" and extended to the level of groupoids.

Finally Chapter 6 contains the proofs of various facts that we use in the main body of the text but that would have disturbed the main line of argument there.

More detailed information can be found in the introduction to the individual Chapters where we also give references to the literature.

Zusammenfassung

Koisotrope Untermannigfaltigkeiten bilden eine natürliche Klasse von Unterobjekten von Poisson Mannigfaltigkeiten; so liefern etwa Lagrangesche Untermannigfaltigkeiten von symplektischen Mannigfaltigkeiten, Hyperflächen, Lie Unteralgebren und Graphen von Poisson-Abbildungen Beispiele von koisotroper Untermannigfaltigkeiten. Ausserdem hat sich in herausgestellt, dass koisotrope Untermannigfaltigkeiten bei der Untersuchung gewisser physikalischer Objekte – genannt "branes" – in der topologischen Stringtheorie ([**KO**]) beziehungsweise im Poisson Sigma Modell ([**CF**]) eine wichtige Rolle spielen. Zu jeder koisotropen Untermannigfaltigkeit gehört eine Blätterung und eine gewisse Quotientenalgebra, welche die Struktur einer Poisson Algebra trägt. Es sind zwei Koketten-Komplexe bekannt, deren nullte Kohomologien isomorph zu der Quotientenalgebra sind: der Lie Algebroid Komplex (siehe [**Mc**]) und der BFV-Komplex (siehe [**BF**] und [**BV**]). Die Kohomologien der beiden Komplexe stimmt in allen Graden miteinerander überein.

In [**OP**] und [**CF**] wurde der Lie Algebroid Komplex mit höheren Operationen ausgestattet, welche die Poisson Klammer auf der Quotientenalgebra kodieren. Diese höheren Operationen sind nicht natürlich und hängen von gewissen Wahlen ab. Wir zeigen, dass unterschiedliche Wahlen zu isomorphen Strukturen führen. Ausserdem ist der BFV-Komplex mit der Struktur einer graduierten Poisson Algebra ausgestattet, welche auch die Poisson Klammer auf der Quotientenalgebra kodiert. Es stellt sich die Frage, inwieweit der Lie Algebroid Komplex ausgestattet mit den höheren Operationen einerseits und die algebraische Struktur auf dem BFV-Komplex andererseits zusammenhängen. Wir beantworten diese Frage, indem wir beweisen, dass die beiden Strukturen "isomorph bis auf Homotopie" – oder L_∞ quasi-isomorph – sind.

Obwohl die beiden Strukturen also eng miteinander verbunden sind, gibt es subtile Unterschiede zwischen ihnen. Während der Lie Algebroid Komplex mit den höheren Operationen nur Informationen über die Jets der Poisson Struktur auf der gegebenen koisotropen Untermannigfaltigkeit beinhaltet, liefert der BFV-Komplex eine Invariante, die vom Keim der Poisson Struktur abhängt. Das hat drastische Konsequenzen für die assozierten glatten Deformationsprobleme: es ist möglich, kleine Deformationen der koisotropen Untermannigfaltigkeit mit Hilfe des BFV-Komplexes zu verstehen – während das im Allgemeinen nicht mit Hilfe des Lie Algebroid Komplex und dessen höheren Operationen möglich ist.

Sei L eine Lagrangsche Untermannigfaltigkeit einer symplektischen Mannigfaltigkeit. Wie bereits bemerkt sind Lagrangsche Untermannigfaltigkeiten Beispiele von koisotropen Untermannigfaltigkeiten. Es ist bekannt, dass der Raum der Lagrangschen Untermannigfaltigkeiten nahe L durch den de Rham Komplex von L beschrieben wird. Ausserdem wird der Modulraum der Lagrangschen Untermannigfaltigkeiten nahe L durch $H^1(L, \mathbb{R})$ beschrieben. Man kann versuchen, diese Aussagen in zwei Richtungen zu erweitern. Einerseits kann man versuchen, globale Effekte zu verstehen. Hierbei verwendet man zumeist pseudo-holomorphe

Objekte. Diese Idee geht auf Gromov ([**Gr**]) und Floer ([**Fl**]) zurück und wurde in [**FOOO**] zu einer vollständigen Theorie entwickelt.

Andererseits kann man versuchen, das lokale Deformationsproblem für Untermannigfaltigkeiten, die allgemeiner als Lagrangsche Untermannigfaltigkeiten sind, zu verstehen. Das Deformationsproblem koisotroper Untermannigfaltigkeiten von symplektischen Mannigfaltigkeiten wurde in [**OP**] untersucht und eine Beschreibung von kleinen Deformationen mit Hilfe des Lie Algebroid Komplexes und seinen höheren Operationen wurde geliefert. Allerdings gilt diese Beziehung zwischen dem Deformationsproblem koisotroper Untermannigfaltigkeiten und dem Lie Algebroid Komplex mit höheren Operationen nicht in der Kategorie der Poisson Mannigfaltigkeiten.

Wir beschreiben die Menge der koisotropen Untermannigfaltigkeiten nahe einer vorgegeben mit Hilfe des BFV-Komplexes – diese Beschreibung ist sogar für koisotrope Untermannigfaltigkeiten von Poisson Mannigfaltigkeiten gültig. Ausserdem wird eine "Kategorifizierung" des Modulraums von koisotropen Untermannigfaltigkeiten nahe einer vorgegebene eingeführt. Dies liefert ein gewisses Groupoid. Mit Hilfe des BFV-Komplexes wird ein anderes Groupoid konstruiert. Es gibt einen surjektiven Morphismus von diesem zweiten Groupoid in das Groupoid, das die kleinen Deformationen der koisotropen Untermannigfaltigkeit beschreibt. Der Kern des Morphismuses kann mit Hilfe des BFV-Komplexes einfach charakterisiert werden und als Korollar erhälten wir eine Bijektion zwischen dem Modulraum der kleinen Deformationen und dem Modulraum gewisser Maurer–Cartan Elemente des BFV-Komplexes.

Die Verbindung zwischen den beiden oben erwähnten Groupoiden kann auf zwei Arten interpretiert werden. Erstens liefert sie einen neuen Zugang zum Deformationsproblem koisotroper Untermannigfaltigkeiten. Zweitens hilft sie, den geometrischen Inhalt des BFV-Komplexes zu verstehen. Hier ist anzumerken, dass der BFV-Komplex gewöhnlicherweise als Startpunkt für die Konstruktion einer (geometrischen oder Deformations-) Quantisierung von koisotropen Untermannigfaltigkeiten betrachtet wird – ein Problem, dass nicht direkt mit dem geometrischen Deformationsproblem zu tun hat.

Für eine detailliertere Beschreibung der Dissertation verweisen wir auf die Einleitung, siehe Kapitel 1.

Teile des hier beschriebenen Ergebnisse sind in [**Sch1**], [**Sch2**] und [**Sch3**] sowie der gemeinsamen Arbeit [**CS**] mit Cattaneo erschienen.

Die Dissertation ist wie folgt aufgebaut: Kapitel 1 enthält eine Einführung in die Dissertation. Wir erkären kurz die wichtisten Konzepte – Poisson Mannigfaltigkeiten, Hamiltonische Diffeomorphismen und koisotropic Untermannigfaltigkeiten – und formulieren das Hauptproblem, nämlich eine Beschreibung der Menge der koisotropen Untermannigfaltigkeiten nahe einer gegebenen zu finden. An Beispielen demonstrieren wir die Verbindung zum Lie Algebroid Komplex und zu den höhreren Operationen, die man auf dem Lie Algebroid Komplex konstruieren kann. Beide Strukturen enhalten wichtige Information, aber reichen nicht aus, das Hauptproblem zu lösen. Schlussendlich geben wir eine grobe Beschreibung des BFV-Komplexes und erkären, wie man mit seiner Hilfe das Hauptproblem lösen kann.

In Kapitel 2 wird das nötige Hintergrundmaterial erläutert. Insbesondere erinnern wir an die grundlegenden Definitionen im Zusammenhang mit L_∞-Algebren (Abschnitt 1), erkären den Transfer von L_∞-Algebren entlang von Kontraktionen (Abschnitt 2), beschreiben den "higher derived brackets"-Formalismus (Abschnitt 3) und füheren Teile der Theorie der glatten graduierten Mannigfaltigkeiten ein (Abschnitt 4).

In Kapitel 3 werde grundlegenden Tatsachen aus der Poisson Geometrie behandelt (Abschnitt 1), koisotrope Untermannigfaltigkeiten eingeführt (Abschnitt 2) und die Konstruktion der höhren Operationen auf dem Lie Algebroid Komplex erklärt.

Kapitel 4 beinhaltet eine Einführung in die Konstruktion des BFV-Komplexes sowie einige seiner Eigenschaften. Als Vorbereitung konstruieren wir einen gewissen L_∞ Quasi-isomorphismus vom Raum der Multivektorfelder auf einer Mannigfaltigkeit M in den Raum der Multivektorfelder auf einer gewissen glatten graduierten Mannigfaltigkeit über M. In Abschnitt 2 verwenden wir diesen L_∞ Quasi-Isomorphismus für eine konzeptuelle Konstruktion der BFV-Klammer. Ausserdem erläutern wir die Konstruktion eines Differentials auf dem BFV-Komplex. Dieses Differential und die BFV-Klammer statten den BFV-Komplex mit der Struktur einer differentiell graduierten Poisson Algebra aus. Die Abhängigkeit dieser algebraischen Struktur von den Wahlen, welche in ihrer Konstruktion involviert sind, wird geklärt. In Abschnitt 3 wird die Verbindung zwischen dem BFV-Komplex einerseits und dem Lie Algebroid Komplex mit höheren Operationen andererseits hergestellt (Theorem 3.6).

Der geometrische Inhalt des BFV-Komplexes wird in Kapitel 5 eingehender untersucht. Der BFV-Komplex liefert eine gewisse Gleichung – Maurer–Cartan Gleichung genannt – und wir untersuchen die Menge der Lösungen dieser Gleichung in Abschnitt 1. Es zeigt sich, dass es einen Zusammenhang zwischen dieser Menge und der Menge der sogenannten koisotropen Schnitte gibt (Theorem 1.13). In Abschnitt 2 beziehen wir die inneren Symmetrien der Poisson Mannigfaltigkeit und der differentiall graduierten Poisson Algebra Struktur auf dem BFV-Komplex in unsere Untersuchungen mit ein. Dies führt zu Theorem 2.25 in dem der Modulraum von koisotropen Schnitten mit dem Modulraum gewisser Maurer–Cartan Elemente identifiziert wird. In Abschnitt 3 wird dieser Zusammenhang "kategorifiziert" und auf Groupoide ausgedehnt.

Kapitel 6 beinhaltet schlussendlich die Beweise verschiedener Aussagen, welche wir im Hauptteil des Textes anwenden, aber deren Aufnahme in den Hauptteil unsere Argumentationsliene zu sehr gestört hätte.

Detaillierte Angaben finden sich in der Einleitung der unterschiedlichen Kapitel, wo wir auch Verweise zur bestehenden Literatur geben.

Acknowledgements

First and foremost I thank Alberto Cattaneo for providing me the opportunity to work on mathematics over the last years, his generosity in sharing ideas and insights, his continuing interest in my work and his overall support. I am very grateful that I had the chance to learn from many mathematicians who shared their knowledge and wisdom with me. In particular my communication with Peter Michor, James Stasheff and Alan Weinstein was invaluable. Furthermore I thank Marco Zambon and Thomas Preu for discussing mathematics with me – it was often a crucial point in the transition from a vague idea to an actual proof.

Moreover I am especially happy to express my thankfullness for all the time I could share with my parents, my brothers, Daniela Liebethat, Stephan Peischl and Thomas Preu. Last but not least I want to thank Günter Czetina for introducing me to the beauty of mathematics and physics.

Contents

CHAPTER 1

Introduction

1. Geometric Background

This thesis is about algebraic structures – and their geometric content – associated to *coisotropic submanifolds* of *Poisson manifolds*.

A *Poisson manifold* is a manifold M equipped with an additional structure, a *Poisson bivector field* Π. Poisson bivector fields are sections of $\wedge^2 TM$ that satisfy a certain integrability condition. One way to express this condition is to require that the *Poisson bracket* associated to Π via

$$\{\cdot,\cdot\}_\Pi : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad (f,g) \mapsto < \pi, d_{DR}f \wedge d_{DR}g >$$

equips the algebra of smooth function $\mathcal{C}^\infty(M)$ with the structure of a Lie algebra. This means that $\{\cdot,\cdot\}_\Pi$ is bilinear, skew-symmetric and satisfies the *Jacobi identity*, i.e.

$$\{f,\{g,h\}_\Pi\}_\Pi = \{\{f,g\}_\Pi,h\}_\Pi + \{g,\{f,h\}_\Pi\}_\Pi$$

holds for all smooth functions f, g and h.

The following list of Poisson bivector fields and Poisson manifolds respectively is intended to give the reader a flavor of what kind of objects live in the "Poisson world":

(a) the zero section $0 \in \Gamma(\wedge^2 TM)$ is always a Poisson bivector field on M,
(b) let Σ be a two dimensional manifold; any section of $\wedge^2 T\Sigma$ is a Poisson bivector field,
(c) let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$; the linear dual $\mathfrak{g}^*$ carries a natural linear Poisson bivector field which encodes the Lie bracket on $\mathfrak{g}$,
(d) any symplectic manifold (M,ω) yields a Poisson manifold.

Coisotropic submanifolds constitute a natural class of submanifolds of Poisson manifolds. Let (M,Π) be a manifold equipped with a Poisson bivector field Π. The *vanishing ideal* of a submanifold S of M is the multiplicative ideal

$$\mathcal{I}(S) := \{f \in \mathcal{C}^\infty(M) : f|_S = 0\}$$

of $\mathcal{C}^\infty(M)$. The submanifold S is *coisotropic* if and only if the vanishing ideal of S in M is a Lie subalgebra of $(\mathcal{C}^\infty(M), \{\cdot,\cdot\}_\Pi)$, i.e.

$$\{\mathcal{I}(S), \mathcal{I}(S)\}_\Pi \subset \mathcal{I}(S).$$

Observe that the condition on the ideal $\mathcal{I}(S)$ also arises in the theory of integrable Hamiltonian systems: assume we are given a system of constants of motions $\{f_1, \dots, f_n\}$ which are in involution. Then the ideal $\mathcal{I}(f_1, \dots, f_n)$ generated by $\{f_1, \dots, f_n\}$ satisfies

$$\{\mathcal{I}(f_1, \dots, f_n), \mathcal{I}(f_1, \dots, f_n)\}_\Pi \subset \mathcal{I}(f_1, \dots, f_n).$$

It follows that whenever certain non-degeneracy conditions are satisfied, the zero set of $\{f_1, \dots, f_n\}$ is a coisotropic submanifold.

Other examples of coisotropic submanifolds are listed below:

(a) any submanifold of $(M, 0)$ is coisotropic,
(b) any open subset of a Poisson manifold (M, Π) is coisotropic – in particular that applies to M itself,
(c) any codimension 1 submanifold of a Poisson manifold (M, Π) is coisotropic,
(d) a point $x \in (M, \Pi)$ is coisotropic if and only if $\Pi_x = 0$,
(e) let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$; a linear subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ if and only if the annihilator $\mathfrak{h}^\circ$ of $\mathfrak{h}$ is a coisotropic submanifold of $\mathfrak{g}^*$,
(f) Lagrangian submanifolds of symplectic manifolds are always coisotropic submanifolds.

Given a Poisson manifold (M, Π), one might wonder what the space of all coisotropic submanifolds contained in (M, Π) looks like. To render this problem more tractable, we will restrict our attention to a small part of this question. First, we are only interested in coisotropic submanifolds "near" a given one. More technically speaking, this amounts to "linearizing" M near S, i.e. we choose an embedding of the normal bundle of S into M and assume from now on that M is the total space of a vector bundle $E \to S$. Second, we restrict our attention to a special class of submanifolds of M, namely those which arise as graphs of sections of E. A section $\mu \in \Gamma(E)$ is called *coisotropic* if its graph is a coisotropic submanifold of (E, Π) and we denote the set of coisotropic sections by $\mathcal{C}(E, \Pi)$. Our initial question simplifies to:

Given (E, Π)*, what is a good way to describe* $\mathcal{C}(E, \Pi)$ *and what are its properties?*

The above considerations do not account for the important fact that every Poisson manifold comes along with a special group of symmetries, the group of *Hamiltonian diffeomorphisms*. These are diffeomorphisms arising as flows of *Hamiltonian vector fields*: If $\{\cdot,\cdot\}_\Pi$ is the Poisson bracket corresponding to a Poisson bivector field Π on M, every smooth function f yields a derivation D_f of $\mathcal{C}^\infty(M)$ via

$$D_f : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad D_f(g) := \{f, g\}_\Pi.$$

It is well-known that derivations of $\mathcal{C}^\infty(M)$ are in one-to-one correspondence with vector fields on M. The vector field corresponding to D_f is the *Hamiltonian vector field* associated to f.

Hamiltonian diffeomorphisms map coisotropic submanifolds to coisotropic submanifolds and hence the group of Hamiltonian diffeomorphisms acts on the set of all coisotropic submanifolds. From the geometric point of view two coisotropic submanifolds should be considered equivalent if there is a Hamiltonian diffeomorphisms mapping one to the other, i.e. one should try to understand the quotient of the set of coisotropic submanifolds under the action of Hamiltonian diffeomorphisms. If we translate this into our framework, we obtain an equivalence relation $\sim_H$ on the set of coisotropic sections $\mathcal{C}(E,\Pi)$. We denote the set of equivalence classes by $\mathcal{M}(E,\Pi)$ and call it the *modular space of coisotropic sections.* We arrive at a new problem:

Given (E,Π)*, what is a good way to describe* $\mathcal{M}(E,\Pi)$ *and what are its properties?*

2. Some Examples

First we consider two simple examples of coisotropic submanifolds of Poisson manifolds where the set of coisotropic sections and the corresponding moduli space can be easily computed.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. The origin is always a coisotropic submanifold of $\mathfrak{g}^*$ and $\mathfrak{g}^*$ is already a vector bundle over $\{0\}$ since $\mathfrak{g}^*$ is a vector space. As remarked before, a point $\xi \in \mathfrak{g}^*$ is a coisotropic submanifold if and only if the Poisson bivector field vanishes at ξ. It is easy to check this is the case if and only if $\xi \in \mathfrak{g}^*$ annihilates all elements of the form $[x,y]$ for x and y in $\mathfrak{g}$. This means that ξ is an element of the annihilator $([\mathfrak{g},\mathfrak{g}])^\circ$ of $[\mathfrak{g},\mathfrak{g}]$. Consequently we obtain

$$\mathcal{C}(\mathfrak{g}^* \to \{0\}, \Pi_{\mathfrak{g}^*}) = ([\mathfrak{g},\mathfrak{g}])^\circ.$$

Vanishing of the Poisson bivector field at a point y implies that all Hamiltonian vector fields vanish at y too. Hence every Hamiltonian diffeomorphism leaves such points invariant and consequently points that are coisotropic submanifolds cannot be moved by Hamiltonian diffeomorphisms. So one obtains

$$\mathcal{M}(\mathfrak{g}^* \to \{0\}, \Pi_{\mathfrak{g}^*}) = ([\mathfrak{g},\mathfrak{g}])^\circ.$$

Another basic example is the x-axis $\mathbb{R} \times \{0\}$ as a coisotropic submanifold of $(\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. Recall that in dimension 2 every bivector field is Poisson and that every codimension 1 submanifold is automatically coisotropic. We consider $\mathbb{R}^2$ as a vector bundle over $\mathbb{R} \times \{0\}$ and sections of that bundle are just smooth functions on $\mathbb{R} \times \{0\} \cong \mathbb{R}$. Since the graph of such a function is again of codimension 1 and hence coisotropic, we obtain

$$\mathcal{C}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \mathcal{C}^\infty(\mathbb{R}).$$

Furthermore it turns out that any two graphs of smooth functions can be connected by a Hamiltonian diffeomorphism. In fact, given two functions f and g it suffices to consider the Hamiltonian vector field generated by the function

$$H : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \int_0^x (f - g)(t)dt.$$

This implies

$$\mathcal{M}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \{*\}.$$

Remarkably, in both examples the set of coisotropic sections and the moduli space of coisotropic sections are tightly related to certain *cochain complexes*. In the case of $\{0\} \hookrightarrow \mathfrak{g}^*$ the relevant complex is the *Chevalley–Eilenberg complex* of the Lie algebra $\mathfrak{g}$ with values in the trivial $\mathfrak{g}$-module $\mathbb{R}$

$$\mathbb{R} \to \mathfrak{g}^* \to \wedge^2\mathfrak{g}^* \to \cdots .$$

Observe that

- the kernel of $\mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is exactly $([\mathfrak{g}, \mathfrak{g}])^\circ$, the annihilator of $[\mathfrak{g}, \mathfrak{g}]$ and
- the first cohomology group $H^1_{CE}(\mathfrak{g}, \mathbb{R})$ is also equal to $([\mathfrak{g}, \mathfrak{g}])^\circ$.

For

$$\mathbb{R} \times \{0\} \hookrightarrow (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x})$$

one considers the de Rham complex of $\mathbb{R}$ given by

$$\mathcal{C}^\infty(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \cong \mathcal{C}^\infty(\mathbb{R}).$$

One can compute easily that

- the kernel of $\Omega^1(\mathbb{R}) \xrightarrow{d} (\Omega^2(\mathbb{R}) = \{0\})$ is isomorphic to $\mathcal{C}^\infty(\mathbb{R})$ and
- the first cohomology group $H^1(\mathbb{R}, \mathbb{R})$ is isomorphic to $\{0\}$.

Observe that in both examples

- the set of coisotropic sections coincides with the space of closed elements of degree 1 and
- the moduli space of coisotropic sections coincides with the first cohomology group.

One is naturally led to wonder:

Given (E, Π), is there a complex $(C^\bullet, d)$ such that

- *the set of coisotropic sections* $\mathcal{C}(E,\Pi)$ *coincides with the kernel of* $C^1 \xrightarrow{d} C^2$ *and*
- *the moduli space of coisotropic sections* $\mathcal{M}(E,\Pi)$ *coincides with* $H^1(C,d)$*?*

It is well-known that this question can be answered affirmatively in the case of *Lagrangian submanifolds* of *symplectic manifolds*. A symplectic manifold is a manifold M together with a closed two form $\omega \in \Omega^2(M)$ such that the vector bundle map

$$\omega^\# : TM \to T^*M$$

given by contraction is an isomorphism. A submanifold L of M is *Lagrangian* if the pull back of ω to L vanishes and the dimension of L is half the dimension of M. It turns out that symplectic manifolds and their Lagrangian submanifolds can be interpreted as special cases of Poisson manifolds and their coisotropic submanifolds. The most basic example of a Lagrangian submanifold of a symplectic manifold corresponds to

$$\mathbb{R} \times \{0\} \hookrightarrow (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}).$$

The calculations of the set of coisotropic sections and of the associated moduli space can be generalized vastly with the help of the Darboux–Weinstein Theorem which implies that it suffices to consider $L \hookrightarrow (T^*L, \omega_{\mathrm{can}})$ where ω_{can} is a universal symplectic structure on the cotangent bundle T^*L. One obtains

PROPOSITION 2.1. (a) *A graph of a section of* $T^*L \to L$ *is coisotropic if and only if the section – seen as a one-form on* L *– is closed with respect to the de Rham differential. Consequently*

$$\mathcal{C}(T^*L \to L, \omega_{\mathrm{can}}) = \{\mu \in \Omega^1 : d_{DR}\mu = 0\}.$$

(b) *Let* μ *and* ν *be two one-forms on* L *which are closed with respect to the de Rham differential. Then the graphs of* μ *and* ν *are equivalent (with respect to* $\sim_H$*) if and only if* $[\mu] = [\nu] \in H^1(L,\mathbb{R})$*. Consequently*

$$\mathcal{M}(T^*L \to L, \omega_{\mathrm{can}}) = H^1(L,\mathbb{R}).$$

A proof can be found in Chapter 5, Section 2.

Thus, in case we are considering Lagrangian submanifolds of symplectic manifolds, the de Rham complex is connected to the set of coisotropic sections and the moduli space of coisotropic sections in the predicted manner. One is tempted to suspect an analogous solution for arbitrary coisotropic submanifolds of Poisson manifolds, so the question arises:

What is the right replacement of the de Rham complex for an arbitrary coisotropic submanifold?

There is a well-known complex associated to every coisotropic submanifold S of a Poisson manifold, its *Lie algebroid complex*. Assume that the Poisson manifold under consideration is the total space of a vector bundle $E \to S$ over the coisotropic submanifold S. Then the graded vector space underlying the Lie algebroid complex of S is $\Gamma(\wedge E)$. For a proper definition of the differential we refer the reader to Chapter 3, Section 2. Let us just make two remarks about this complex:

- for a Lagrangian submanifold of a symplectic manifold, the Lie algebroid complex is isomorphic to the de Rham complex of the Lagrangian submanifold,
- for $\{0\} \hookrightarrow \mathfrak{g}^*$ – $\mathfrak{g}$ a finite dimensional Lie algebra over $\mathbb{R}$ – the Lie algebroid complex is the Chevalley–Eilenberg complex of $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $\mathbb{R}$.

Hence in both examples we considered so far – the origin inside $\mathfrak{g}^*$ and a Lagrangian submanifold of a symplectic manifold – the Lie algebroid complex allows us to recover $\mathcal{C}(E,\Pi)$ and $\mathcal{M}(E,\Pi)$ as described above. This motivates the question:

Can one always recover $\mathcal{C}(E,\Pi)$ and $\mathcal{M}(E,\Pi)$ from the Lie algebroid complex?

The next example shows that this is not the case: Consider the x-axis $\mathbb{R}\times\{0\}$ inside $(\mathbb{R}^2,(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y})$. By dimensional reasons this is a coisotropic submanifold and one can compute

- $\mathcal{C}(\mathbb{R}^2 \to \mathbb{R}\times\{0\},(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}) \cong \mathcal{C}^\infty(\mathbb{R})$,
- $\mathcal{M}(\mathbb{R}^2 \to \mathbb{R}\times\{0\},(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}) \cong \{+\}\coprod\mathbb{R}\coprod\{-\}$.

The geometric meaning of $\mathcal{M}(\mathbb{R}^2 \to \mathbb{R}\times\{0\},(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y})$ is the following: Let f be a smooth function on $\mathbb{R}$ and denote its equivalence class under $\sim_H$ by $[f]$. Now one distinguishes three cases: either $f(0)<0$, or $f(0)>0$ or $f(0)=0$. If $f(0)<0$, the graph of f can be mapped to any other graph of a function g with $g(0)<0$ by a Hamiltonian diffeomorphism. We denote the corresponding equivalence class of functions by "$-$". In case $f(0)>0$, the graph of f is equivalent to all the graphs of functions whose value at zero is bigger than 0. We denote the corresponding equivalence class by "$+$". If $f(0)=0$ one can show that f is equivalent to an other function g with $g(0)=0$ if and only if $f'(0)=g'(0)$. So we can identify equivalence classes of functions which vanish at the origin with their first derivative at the origin. All this can be summarized in the following commutative triangle

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \{+\}\coprod\mathbb{R}\coprod\{-\} \\
& \mathcal{M}(\mathbb{R}^2 \to \mathbb{R}\times\{0\},(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}). \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Here p denotes the natural projection which maps functions to their equivalence classes and Φ is given by

$$f \mapsto \begin{cases} + & f(0) > 0, \\ f'(0) & f(0) = 0, \\ - & f(0) < 0. \end{cases}$$

The Lie algebroid complex of $\mathbb{R} \times \{0\} \hookrightarrow (\mathbb{R}^2, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$ is isomorphic to

$$0 \to \mathcal{C}^\infty(\mathbb{R})[0] \xrightarrow{x^2(-)} \mathcal{C}^\infty(\mathbb{R})[-1] \to 0.$$

The numbers inside $[-]$ refer to the fact that the first copy of the algebra of smooth functions lives in degree 0 while the second one lives in degree 1. Consequently

- the kernel of $\mathcal{C}^\infty(\mathbb{R})[-1] \to 0$ is $\mathcal{C}^\infty(\mathbb{R})$,
- the first cohomology group is isomorphic to $\mathbb{R}^2$ and the isomorphism is induced by

$$\mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}^2, \quad f \mapsto (f(0), f'(0)).$$

Thus, do not recover $\mathcal{M}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \{+\} \coprod \mathbb{R} \coprod \{-\}$. We remark that the Lie algebroid complex of $\mathbb{R} \times \{0\}$ as a coisotropic submanifold of $(\mathbb{R}^2, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$ coincides with the Lie algebroid complex of $\mathbb{R} \times \{0\}$ as a coisotropic submanifold of $(\mathbb{R}^2, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. One can verify that in the latter case the Lie algebroid complex actually yields the right answer since

- $\mathcal{C}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, x^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \mathcal{C}^\infty(\mathbb{R})$ and
- $\mathcal{M}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, x^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \mathbb{R}^2$, where the isomorphism is induced from

$$\mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}^2, \quad f \mapsto (f(0), f'(0)).$$

Furthermore observe that $\mathcal{M}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$ cannot be equipped with the structure of a vector space such that the projection from

$$\mathcal{C}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \mathcal{C}^\infty(\mathbb{R})$$

becomes a linear map. This implies that there is no way to obtain

$$\mathcal{M}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \{+\} \coprod \mathbb{R} \coprod \{-\}$$

using constructions from linear algebra alone – some non-linearity has to be added. An appropriate "non-linear enrichment" of the Lie algebroid complex can be achieved and is known as the *homotopy Lie algebroid*.

3. The homotopy Lie Algebroid

In the previous Section we saw that it is not always possible to recover the set of coisotropic sections and the corresponding moduli space from the Lie algebroid complex of the coisotropic submanifold. In [**OP**] *higher order operations* on the Lie algebroid complex for coisotropic submanifolds of symplectic manifolds were defined and applications to the *formal deformation problem* of coisotropic submanifolds were given. The construction of these operations was extended to arbitrary submanifolds of Poisson manifolds in [**CF**] and connections to the *deformation quantization* of coisotropic submanifolds were investigated. We will restrict our exposition to the case of coisotropic submanifolds of Poisson manifolds in the following. The Lie algebroid complex together with the higher order operation is referred to as the *homotopy Lie algebroid* of the coisotropic submanifold.

So what are these higher order operations? Let S be a coisotropic submanifold of (E,Π) and assume that $E \to S$ is a vector bundle. Oh, Park and Cattaneo, Felder defined an infinite sequence of multi linear maps

$$\lambda_n : \Gamma(\wedge E) \times \cdots \times \Gamma(\wedge E) \to \Gamma(\wedge E)[2-n],$$

i.e. n sections $(\xi_1,\dots,\xi_n)$ of $\wedge E$ get mapped to a section of $\wedge E$. If the sum of the degrees of the individual ξ_ks is m, the section $\lambda_n(\xi_1,\dots,\xi_n)$ will be of degree $m+2-n$. We refer the reader to Chapter 3, Section 2 for a proper definition of the higher order operations.

Let us give a list of some properties of the higher order operations $(\lambda_n)_{n\geq 1}$:

- for $n=1$ we obtain the coboundary operator of the Lie algebroid complex of S,
- the family $(\lambda_n)_{n\geq 1}$ satisfies a family of quadratic relations that makes it into an L_∞*-algebra* on $\Gamma(\wedge E)$; in particular:
  - the first quadratic relation is $\lambda_1 \circ \lambda_1 = 0$,
  - the second one implies that the coboundary operator of the Lie algebroid complex is compatible with λ_2 (it is a graded derivation of λ_2),
  - the third one implies that λ_2 is a graded Lie bracket on $\Gamma(\wedge E)$ up to a violation of the graded Jacobi identity by a λ_1-exact term,
- one can define the higher order operations also without assuming the Poisson manifold (E,Π) to be a vector bundle over S; then the higher order operations are not uniquely defined but their isomorphism class is (see [**CS**]).

Let us describe what these higher order operations amount to in the case of

$$\mathbb{R}\times\{0\} \hookrightarrow (\mathbb{R}^2, (x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}).$$

Only a single operation

$$\begin{array}{rcl}\lambda_3 : \mathcal{C}^\infty(\mathbb{R})[0] \times \mathcal{C}^\infty(\mathbb{R})[-1] \times \mathcal{C}^\infty(\mathbb{R})[-1] & \to & \mathcal{C}^\infty(\mathbb{R})[-1] \\ (f, g, h) & \mapsto & \dfrac{df}{dx} gh\end{array}$$

is added to the differential λ_1 on $(\mathcal{C}^\infty(\mathbb{R})[0] \to \mathcal{C}^\infty(\mathbb{R})[-1])$. One is naturally led to ask

What is the appropriate replacement of

- *closed elements of degree* 1*,*
- *the first cohomology group*

in the presence of higher order operations?

Let $(C^\bullet, d)$ be a complex enriched by higher order operations $(\lambda_k)_{k \geq 1}$ (we assume $d = \lambda_1$). It turns out that the set of closed elements of degree 1 should be replaced by the set of all elements of degree 1 that satisfy the *Maurer–Cartan equation*

$$\sum_{k \geq 1} \frac{1}{k!} \lambda_k(\alpha, \ldots, \alpha) = 0.$$

Every such element α is called a *Maurer–Cartan element*. Moreover the action of C^0 on the set of closed element of degree 1 given by

$$\begin{array}{rcl}C^0 \times \ker\left(d : C^1 \to C^2\right) & \to & \ker\left(d : C^1 \to C^2\right) \\ (v, x) & \mapsto & x + d(v)\end{array}$$

is replaced by the *gauge action*. The gauge action induces an equivalence-relation – known as the *gauge equivalence* – on the set of Maurer–Cartan elements. The first cohomology group $H^1(C^\bullet, d)$ is replaced by the set of equivalence classes of Maurer–Cartan elements with respect to the gauge equivalence.

If we apply this abstract machinery to the Lie algebroid complex of

$$\mathbb{R} \times \{0\} \hookrightarrow (\mathbb{R}^2, (x^2 + y^2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$$

together with the operation λ_3 that was spelled out above, we arrive at the following picture: The set of Maurer–Cartan elements is $\mathcal{C}^\infty(\mathbb{R})[-1]$ and two functions g and h are gauge equivalent if and only if there are two smooth functions f and α on $[0, 1] \times \mathbb{R}$ such that

- $f_0 := f|_{\{0\} \times \mathbb{R}} = 0$,
- $\alpha_0 = g$, $\alpha_1 = h$,
- for all $t \in [0, 1]$ and $x \in \mathbb{R}$ the following differential equation holds:

$$\frac{\partial \alpha(t, x)}{\partial t} = \left(x^2 + \frac{1}{2}\alpha^2(t, x)\right) \frac{\partial f(t, x)}{\partial x}.$$

Here the higher order operation λ_3 causes the term $\frac{1}{2}\alpha^2(t,x)$ to appear in the above differential equation. This makes the gauge equivalence non-linear. Without that term, the equivalence relation could be reduced to

$$g \sim h \quad \Leftrightarrow \quad h - g = x^2 \frac{df}{dx}$$

and we would recover the first cohomology group of the Lie algebroid complex. However, if one takes the higher order operations into account, the result is

$$(\{\text{Maurer–Cartan elements}\}/\text{gauge equivalence}) \cong \{+\} \coprod \mathbb{R} \coprod \{-\}$$

and the isomorphism is induced by

$$\mathcal{C}^\infty(\mathbb{R}) \to \{+\} \coprod \mathbb{R} \coprod \{-\}, \quad f \mapsto \begin{cases} + & f(0) > 0, \\ f'(0) & f(0) = 0, \\ - & f(0) < 0. \end{cases}$$

Observe that

$$\{\text{Maurer–Cartan elements}\}/\text{gauge equivalence}$$

coincides with $\mathcal{M}(\mathbb{R}^2 \to \mathbb{R} \times \{0\}, (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$!

This is an indication that the higher order operations on the Lie algebroid complex add the information which is needed to recover the set of coisotropic sections and the moduli space of coisotropic sections respectively.

Can one always recover $\mathcal{C}(E,\Pi)$ and $\mathcal{M}(E,\Pi)$ from the Lie algebroid complex equipped with the higher order operations?

Oh and Park proved that this question can be answered positively for $\mathcal{C}(E,\Pi)$ in case the Poisson bivector field Π comes from a symplectic structure. However in genereal – i.e. including $\mathcal{M}(E,\Pi)$ and arbitrary Poisson bivector fields – the question cannot be answered affirmatively. The problem is that for a coisotropic submanifold S of a Poisson manifold (E,Π), there might be infinitely many $k \geq 1$ such that the higher order operation λ_k is non-trivial. Thus, one needs some kind of *completion* to make sense of the notion of Maurer–Cartan elements and gauge equivalence respectively because both are defined via infinite sums. From an algebraic point of view the natural completion is a *formal* one and this does not take all the relevant geometric information into account. For instance the higher order operations of

$$\{0\} \hookrightarrow (\mathbb{R}^2, \Pi)$$

where

$$\Pi := \begin{cases} 0 & (x,y) = (0,0) \\ e^{-\left(\frac{1}{x^2+y^2}\right)} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} & (x,y) \neq (0,0) \end{cases}$$

all vanish, i.e. the higher order operations cannot distinguish $\{0\} \hookrightarrow (\mathbb{R}^2,\Pi)$ from $\{0\} \hookrightarrow (\mathbb{R}^2, 0)$. Observe that the two Poisson manifolds are quite different: Π is only zero at the origin and so the only point in $\mathbb{R}^2$ that is a coisotropic submanifold is 0. On the other hand, every point is coisotropic as a submanifold of $(\mathbb{R}^2, 0)$.

Although the higher order operations added important information to the Lie algebroid complex of a coisotropic submanifold, we saw that in general this structure is not sufficiently fine to detect all the relevant geometry. It turns out that there is another structure associated to a coisotropic submanifold which overcomes this problem and which we explain next.

4. The BFV-Complex

The *BFV-complex* is an algebraic structure which was originally developed by physicists to handle Hamiltonian systems with complicated symmetries. Its first incarnation goes back to the *BRST-formalism* which was developed by Becchi, Rouet and Stora and independently by Tyutin [**BRST**] – the Hamiltonian version which is relevant for the following discussion was spelled out in detail in [**KSt**]. The Hamiltonian BRST-formalism was extended by Batalin, Fradkin and Vilkovsiky ([**BF**], [**BV**]) to coisotropic submanifolds which are given in terms of constraints. Later on Stasheff ([**Sta2**]) provided an interpretation of their construction using homological algebra. Bordemann and Herbig adapted the BFV-complex to the setting of coisotropic submanifolds of (finite dimensional) Poisson manifolds, see [**B**], [**He**]. Some parts of Bordemann and Herbig's work were given a conceptual interpretation in [**Sch1**]. This led to a clarification of the dependence of the BFV-complex on certain auxiliary data, see [**Sch2**].

To give the reader an idea of the BFV-construction, we will explain the Hamiltonian BRST-formalism following Kostant and Sternberg's work ([**KSt**]). Let (M, π) be a Poisson manifold and $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. Assume that there is a Poisson map $J : M \to \mathfrak{g}^*$, i.e.

$$J^*(\{f, g\}_{\Pi_{\mathfrak{g}^*}}) = \{J^*(f), J^*(g)\}_\Pi$$

holds for all smooth functions f and g on $\mathfrak{g}^*$. Recall that $\{\cdot, \cdot\}_\Pi$ denotes the Poisson bracket associated to the Poisson bivector field Π. Any such map J is called an *equivariant momentum map* and it defines a Lie algebra action of $\mathfrak{g}$ on M, i.e. there is a Lie algebra anti homomorphism

$$\mathfrak{g} \to \Gamma(TM)$$

given by

$$\begin{array}{rcl} \mathfrak{g} \cong (\mathfrak{g}^*)^* \hookrightarrow \mathcal{C}^\infty(\mathfrak{g}^*) \xrightarrow{J^*} \mathcal{C}^\infty(M) & \to & \Gamma(TM) \\ f & \mapsto & X_f. \end{array}$$

Here X_f denotes the Hamiltonian vector field of f.

If one assumes in addition that 0 is a regular value of $J : M \to \mathfrak{g}^*$, the set

$$C := \{x \in M : J(x) = 0\}$$

is a coisotropic submanifold. Moreover the action of $\mathfrak{g}$ on M restricts to an action on C. If the quotient $\underline{C} := C/\mathfrak{g}$ is smooth, it carries a Poisson bivector field inherited from (M, Π). In this case $\underline{C}$ equipped with this Poisson bivector field is called the *reduced phase space*. If the quotient $\underline{C}$ is not smooth, one still obtains a Poisson bracket on the set $(\mathcal{C}^\infty(C))^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant functions on C.

The aim of the Hamiltonian BRST-formalism is to describe the algebra $(\mathcal{C}^\infty(C))^{\mathfrak{g}}$ together with the induced Poisson bracket in a cohomological fashion. To achieve this one combines the *Koszul complex* of the vanishing ideal $\mathcal{I}(C)$ of C in M with the *Chevalley–Eilenberg complex* of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathcal{C}^\infty(M)$ – the module structure corresponds to the Lie algebra action. In more detail, this works as follows:

- The Koszul-complex of $\mathcal{I}(C)$ is $\wedge^{-\bullet}\mathfrak{g} \otimes \mathcal{C}^\infty(M)$ equipped with a coboundary operator
$$d : \wedge^k \mathfrak{g} \otimes \mathcal{C}^\infty(M) \to \wedge^{k-1}\mathfrak{g} \otimes \mathcal{C}^\infty(M)$$
which is defined to be $d(X) = J(X)$ on $\mathfrak{g}$ and is extended as a graded derivation of degree -1 on all of $\wedge^\bullet \mathfrak{g} \otimes \mathcal{C}^\infty(M)$. Due to the regularity condition on J, the cohomology of this complex is concentrated in degree 0 and it is isomorphic to the algebra $\mathcal{C}^\infty(C)$.
- The Chevalley–Eilenberg complex is $\wedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M)$ equipped with a differential δ that takes the Lie bracket on $\mathfrak{g}$ and the action of $\mathfrak{g}$ on M into account.
- It can be checked that the Koszul-differential and the Chevalley–Eilenberg differential commute. So if one defines D on $\wedge^k \mathfrak{g} \otimes \wedge^l \mathfrak{g}^* \otimes \mathcal{C}(M)$ to be $d + (-1)^k \partial$ one obtains a new complex
$$(\wedge^{-\bullet}\mathfrak{g} \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M), D).$$
By a spectral sequence argument the cohomology of total degree 0 of this complex is isomorphic to the algebra $(\mathcal{C}^\infty(C))^{\mathfrak{g}}$.
- The Poisson bracket $\{\cdot,\cdot\}_\Pi$ on $\mathcal{C}^\infty(M)$ extends to $\wedge^{-\bullet}\mathfrak{g} \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M)$ via
$$\{X \otimes \xi \otimes f, Y \otimes \zeta \otimes g\}_\Pi := (-1)^{|\xi||Y|}(X \wedge Y) \otimes (\xi \wedge \zeta) \otimes \{f, g\}_\Pi.$$
Moreover the contraction between $\mathfrak{g}$ and $\mathfrak{g}^*$ can also be extended as a graded biderivation to $\wedge^{-\bullet}\mathfrak{g} \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M)$. We denote the resulting operation by $< \cdot, \cdot >$.
- The sum
$$[[\cdot,\cdot]] := \{\cdot,\cdot\}_\Pi + < \cdot, \cdot >$$
satisfies rules analogous to the ones satisfied by $\{\cdot,\cdot\}_\Pi$ – the only difference is that additional signs appear. Thus the graded algebra $\wedge^{-\bullet}\mathfrak{g} \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M)$ together with $[[\cdot,\cdot]]$ is an example of a *graded Poisson algebra*.

- The crucial point of the whole construction is the fact that the differential D can be written as
$$D(\cdot) = [[\theta, \cdot]]$$
for a certain element θ – called the *BRST-charge* – of
$$\left(\wedge\mathfrak{g}^* \oplus (\mathfrak{g} \otimes \wedge^2\mathfrak{g}^*)\right) \otimes \mathcal{C}^\infty(M).$$
This implies that D and $[[\cdot,\cdot]]$ are compatible, i.e. D is a graded derivation of degree 1 with respect to $[[\cdot,\cdot]]$. Consequently
$$(\wedge^{-\bullet}\mathfrak{g} \otimes \wedge^\bullet\mathfrak{g}^* \otimes \mathcal{C}^\infty(M), D, [[\cdot,\cdot]])$$
is an example of a *differential graded Poisson algebra*.
- Because D and $[[\cdot,\cdot]]$ are compatible, the bracket $[[\cdot,\cdot]]$ descends to the zero cohomology group, which is isomorphic to $(\mathcal{C}^\infty(C))^{\mathfrak{g}}$, and yields an ordinary Poisson bracket there.

It turns out that this procedure can be generalized to arbitrary coisotropic submanifolds S of Poisson manifolds (M,π) – a careful construction is presented in Chapter 4, Section 2 for instance. Again, one obtains a differential graded Poisson algebra
$$(BFV(E,\Pi), D = [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV}).$$
Here Ω corresponds to θ and $[\cdot,\cdot]_{BFV}$ corresponds to $[[\cdot,\cdot]]$.The differential graded Poisson algebra depends on certain auxiliary data. Nevertheless it can be used to construct an invariant of the Poisson manifold, since the dependence on the auxiliary data is well-controlled, see [**Sch2**].

Now that we found a second algebraic structure associated to a coisotropic submanifold S of (E,Π) the following question arises:

What is the connection between the homotopy Lie algebroid – i.e. the Lie algebroid complex equipped with the higher order operations – and the BFV-complex?

It is easy to show that the underlying complexes have isomorphic cohomology, i.e.
$$H^\bullet(BFV(E,\Pi), D) \cong H^\bullet(\Gamma(\wedge E), \lambda_1).$$
If we want to take all the higher order operations on the Lie algebroid complex and the bracket $[\cdot,\cdot]_{BFV}$ into account, the situation is more involved. However, it was proved in [**Sch1**] that the following Theorem holds:

THEOREM 1. *The BFV-complex* $(BFV(E,\Pi), D, [\cdot,\cdot]_{BFV})$ *and the homotopy Lie algebroid* $(\Gamma(\wedge E), (\lambda_n)_{n\geq 1})$ *are* L_∞ *quasi-isomorphic.*

A proof of this Theorem can be found in Chapter 4, Section 3. A few clarifying remarks concerning the Theorem are in order: The BFV-complex and the homotopy Lie algebroid are both objects of the category of L_∞-algebras. Since their underlying graded vector spaces are not isomorphic, they cannot be isomorphic in this category. However, there is a natural homotopy category of L_∞-algebras which is constructed by formally inverting a certain class of morphisms of L_∞-algebras – the L_∞ quasi-isomorphisms. The Theorem asserts that in this homotopy category the BFV-complex and the homotopy Lie algebroid are isomorphic objects. So morally speaking, the BFV-complex and the homotopy Lie algebroid are "isomorphic up to a coherent system of higher homotopies".

5. The Groupoid of coisotropic Sections

Let (M, Π) be a Poisson manifold. Recall that our initial motivation was to find a description of the set of all coisotropic submanifolds of (M, Π). Moreover, we observed that the set of coisotropic submanifolds is acted upon by the group of Hamiltonian diffeomorphisms and it is natural to study the quotient of this group action.

To simplify these problems, we restricted our attention to submanifolds near a fixed coisotropic one. One way to formalize this is

- to choose an embedding of the normal bundle E of S into M and
- only to take graphs of sections of $E \to S$ into account.

In this setting we defined the set of coisotropic sections of $E \to S$, which we denoted by $\mathcal{C}(E, \Pi)$. The action of Hamiltonian diffeomorphisms mentioned above can be encoded in an equivalence relation $\sim_H$ on $\mathcal{C}(E, \Pi)$ and we denoted the corresponding set of equivalence classes by $\mathcal{M}(E, \Pi)$, the moduli space of coisotropic sections. Then we observed that the Lie algebroid complex of S allows us to recover $\mathcal{C}(E, \Pi)$ and $\mathcal{M}(E, \Pi)$ in some cases, but in general the complex alone does not contain enough information. We tried to cure this insufficiency by taking the higher oder operations, which were constructed by Oh, Park and Cattaneo, Felder, into account. In fact, this improves the situation, but there are still examples where one does not obtain the right answer.

In [**Sch3**] a way around this problem was found: it turns out that instead of the homotopy Lie algebroid – i.e. the Lie algebroid complex enriched by higher order operations – one should use the BFV-complex. The precise way in which $\mathcal{C}(E, \Pi)$ and $\mathcal{M}(E, \Pi)$ are related to the BFV-complex is described in Chapter 5. In the following we outline the relevant constructions and the final results.

First of all it is convenient to combine $\mathcal{C}(E, \Pi)$ and $\mathcal{M}(E, \Pi)$ into a single object, known as a *groupoid*. A groupoid is a category all of whose morphisms are invertible. For the purposes we have in mind, it suffices to consider *small* groupoids, i.e. we assume that the objects form a set. Such a groupoid can be described as follows:

- there is a set X, the *set of objects*,

- for all x, y in X, there is a set $\mathrm{Hom}(x,y)$, the *set of homomorphisms* from x to y,
- for all x, y and z in X there is a map
$$\circ : \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \to \mathrm{Hom}(x,z),$$
called the *composition*, such that for all $f \in \mathrm{Hom}(x,y)$, $g \in \mathrm{Hom}(y,z)$ and $h \in \mathrm{Hom}(z,w)$
$$h \circ (g \circ f) = (h \circ g) \circ f$$
holds,
- for every x in X there is $\mathrm{id}_x \in \mathrm{Hom}(x,x)$ such that
  - for all $y \in X$ and $f \in \mathrm{Hom}(x,y)$: $f \circ \mathrm{id}_x = f$ and
  - for all $z \in X$ and $g \in \mathrm{Hom}(z,x)$: $\mathrm{id}_x \circ g = g$,
- for every $f \in \mathrm{Hom}(x,y)$ there is $g \in \mathrm{Hom}(y,x)$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$.

Note that if X contains exactly one element x, all the axioms simplify to the statement that $\mathrm{Hom}(x,x)$ is a group.

One can construct a groupoid $\hat{\mathcal{C}}(E,\Pi)$ with the following properties:

- the set of objects of $\hat{\mathcal{C}}(E,\Pi)$ is the set of coisotropic sections $\mathcal{C}(E,\Pi)$,
- if we identify two objects in $\hat{\mathcal{C}}(E,\Pi)$ whenever there is at least one morphism between them, we obtain the moduli of coisotropic sections $\mathcal{M}(E,\Pi)$.

The morphisms of $\hat{\mathcal{C}}(E,\Pi)$ are obtained by considering *homotopies* of coisotropic sections which are generated by Hamiltonian diffeomorphisms. This means that – given two coisotropic sections μ and ν – we consider smooth one-parameter families of coisotropic sections μ_t such that

- $\mu_0 = \mu$, $\mu_1 = \nu$,
- the graph of μ_t is given by applying a Hamiltonian diffeomorphism ϕ_t to the graph of μ.

To obtain a well-defined composition, one has to mod out homotopies of such homotopies and the morphisms of $\hat{\mathcal{C}}(E,\Pi)$ are homotopy classes of pairs (μ_t, ϕ_t) as described above. Observe that this whole construction is analogous to the construction of the *fundamental groupoid* of a topological space.

Let
$$(BFV(E,\Pi), D = [\Omega, \cdot]_{BFV}, [\cdot,\cdot]_{BFV})$$
be a BFV-complex for the coisotropic submanifold S of (E,Π). This also yields a groupoid, which we denote by $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$:

- objects of $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$ are certain Maurer–Cartan elements of the BFV-complex, i.e. elements β of degree 1 such that

– β satisfied the Maurer–Cartan equation; this means that
$$[\Omega + \beta, \Omega + \beta]_{BFV} = 0$$
holds,

– β satisfied a certain non-degeneracy condition,

we call any any such element β a *geometric Maurer–Cartan element*,

- morphisms of $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ are given in terms of homotopies of geometric Maurer–Cartan elements, i.e. we consider one-parameter family of geometric Maurer–Cartan elements β_t and require that β_0 gets mapped to β_t by a gauge transformation ψ_t,
- again one has to mod out homotopies of such homotopies to obtain a well-defined composition, i.e. morphisms of $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ are homotopy classes of pairs (β_t, ψ_t) as described above.

Now we can state the main result which first appeared in [**Sch3**]:

THEOREM 2. *Let S be a coisotropic submanifold of a Poisson manifold (E, Π) and assume that $E \to S$ is a vector bundle.*

There is a morphism of groupoids
$$\mathcal{L}_{\mathrm{geo}} : \hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi) \to \hat{\mathcal{C}}(E, \Pi)$$
that is surjective on objects and on all homomorphism sets.

Moreover, the kernel of $\mathcal{L}_{\mathrm{geo}}$ is a subgroupoid of $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ that can be intrinscially characterized with the help of the BFV-complex. Hence we obtain a description of the groupoid $\hat{\mathcal{C}}(E, \Pi)$ and consequently of the set of coisotropic sections $\mathcal{C}(E, \Pi)$ and the moduli space of coisotropic sections $\mathcal{M}(E, \Pi)$ in terms of the BFV-complex:

COROLLARY 3. *In the situation of the above Theorem, the following statements hold:*

(a) *The groupoids $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)/\ker(\mathcal{L})$ and $\hat{\mathcal{C}}(E, \Pi)$ are isomorphic. In particular their underlying set of objects are in bijection. Recall that the set of objects of $\hat{\mathcal{C}}(E, \Pi)$ is $\mathcal{C}(E, \Pi)$.*

(c) *Furthermore the set of isomorphisms classes of objects of $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ is isomorphic to the set of isomorphism classes of objects of $\hat{\mathcal{C}}(E, \Pi)$. Note that the latter set is equal to $\mathcal{M}(E, \Pi)$.*

CHAPTER 2

Tools

Most of the material contained in this Chapter is standard. In Section 1 we review the basic definitions concerning L_∞-algebras. Following [**Sta1**] we give the interpretation in terms of coalgebras and codifferentials. Section 2 describes the transfer of L_∞-algebra structures along contraction data. This procedure is well-known to the experts. But since it is one of our main tools throughout Chapter 4 and we could not find an exposition in the literature that really matches what we need for the applications we have in mind, this Section is rather detailed. However [**GL**], [**Me**] and [**KSo**] contain most of the ingredients. Chapter 3 describes the higher derived brackets formalims due to Th. Voronov. It also contains our only really original contribution to the material presented in this Chapter: Theorem 3.7 (which was proved in a slightly more general form in a joint paper with Cattaneo [**CS**]). Finally Section 4 recalls rudiments of the theory of smooth graded manifolds and introduces P_∞-algebras. Readers familiar with the topics mentioned so far should feel free to skip this Chapter.

1. L_∞-algebras

Definition 1.1. Let k be a field and G a group. A *G-graded vector space* V over k is a collection $(V_g)_{g\in G}$ of vector spaces over k. The *homogeneous elements* of degree $g \in G$ of a G-graded vector space V are the elements of V_g. The *h'th suspension* $V[h]$ of a G-graded vector spaces V is the G-graded vector space $(V_{(h\cdot g)})_{g\in G}$.

A *morphism* f from the G-graded vector space V to the G-graded vector space W is a collection of linear maps $(f_g : V_g \to W_g)_{g\in G}$. The *h'th suspension* $f[h] : V[h] \to W[h]$ of $f : V \to W$ is given by the collection of linear maps $(f_{g\cdot h} : V_{g\cdot h} \to W_{g\cdot h})$.

We denote the vector space of morphisms from a G-graded vector space V to a G-graded vector space W by $\mathrm{Hom}(V, W)$. Moreover, $\underline{\mathrm{Hom}}(V, W)$ is the graded vector space whose homogeneous component of degree g is given by $\mathrm{Hom}(V, W[g])$.

Remark 1.2. The class of all G-graded vector spaces over a field k forms a category Vect_k^G and the k'th suspension $[k]$ defines an automorphism of Vect_k^G.

All the examples we will consider are $\mathbb{Z}$-graded vector spaces over $\mathbb{R}$. Whenever we talk about *graded vector spaces* without further specifications this refers to the special case $k = \mathbb{R}$ and $G = \mathbb{Z}$.

DEFINITION 1.3. Let V and W be graded vector spaces. The *tensor product* $V \otimes W$ is the graded vector space given by the collection

$$(V \otimes W)_m := \oplus_{i+j=m} V_i \otimes W_j.$$

Given $f \in \mathrm{Hom}(V, A[|f|])$ and $g \in \mathrm{Hom}(W, B[|g|])$, the *tensor product* $f \otimes g \in \mathrm{Hom}(V \otimes W \to (A \otimes B)[|f| + |g|])$ is determined by setting

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w)$$

on all homogeneous $f \in A$ and $g \in W$.

DEFINITION 1.4. Let V be a graded vector space.

The *tensor algebra* $\mathcal{T}(V)$ is the graded vector space given by the collection

$$(\mathcal{T}(V)_m := \oplus_{k \geq 0} \oplus_{j_1 + \cdots + j_k = m} V_{j_1} \otimes \cdots \otimes V_{j_k})_{m \in \mathbb{Z}}.$$

For $k = 0$ we set the above summand equal to $\mathbb{R}$ (concentrated in degree 0)-

LEMMA 1.5. *Let V be a graded vector space. The* deconcatenation product $\Delta : \mathcal{T}(V) \to \mathcal{T}(V) \otimes \mathcal{T}(V)$ *given by*

$$\Delta(x_1 \otimes \cdots x_k) := \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{(i+1)} \otimes \cdots \otimes x_k)$$

defines the structure of a coassociative coalgebra on the tensor algebra $\mathcal{T}(V)$, i.e. the maps

$$(\Delta \otimes id) \circ \Delta \quad \textit{and} \quad (id \otimes \Delta) \circ \Delta$$

from $\mathcal{T}(V)$ to $\mathcal{T}(V) \otimes \mathcal{T}(V) \otimes \mathcal{T}(V)$ coincide.

PROOF. That is an easy verification. □

REMARK 1.6. Every component $\mathcal{T}(V)_m$ of the tensor algebra of a graded vector space V can be decomposed with respect to the tensor product degree, i.e. every element of $\mathcal{T}(V)_m$ can be uniquely written as a sum of elements of the vector spaces $\mathcal{T}^{(k)}(V)_m := \mathcal{T}(V)_m \cap V^{\otimes k}$ for $k \geq 1$ and $\mathcal{T}^{(0)}(V) = \mathbb{R}$. The vector space $\mathcal{T}^{(k)}(V)_m$ carries two natural actions of the group of permutations Σ_k of a set of k elements. The *even* representation is given by defining

$$\tau \cdot (x_1 \otimes \cdots \otimes x_k) := (-1)^{|x_i||x_{(i+1)}|} x_1 \otimes \cdots \otimes x_{(i+1)} \otimes x_i \otimes \cdots \otimes x_k$$

where τ is the transposition of the i'th and the $(i+1)$'th element. This extends uniquely to an action of Σ_k on $\mathcal{T}^{(k)}(V)_m$. The *odd* representation is given by defining

$$\tau \cdot (x_1 \otimes \cdots \otimes x_k) := -(-1)^{|x_i||x_{(i+1)}|} x_1 \otimes \cdots \otimes x_{(i+1)} \otimes x_i \otimes \cdots \otimes x_k$$

where τ is the transposition of the i'th and the $(i+1)$'th element.

DEFINITION 1.7. Given a graded vector space V, the *graded symmetric algebra* $\mathcal{S}(V)$ of V is the graded vector space given componentwise by the invariants of the even representation on $T(V)_m$. The *graded skew-symmetric algebra* $\Lambda(V)$ is the graded vector space given componentwise by the invariants of the odd representation on $T(V)_m$.

REMARK 1.8. The graded symmetric and the graded skew-symmetric algebra $\mathcal{S}(V)$ and $\Lambda(V)$ come along with an additional degree with is inherited from the degree on $\mathcal{T}(V)$ with respect to $\otimes$. We denote the homogeneous elements with respect to this degree with $\mathcal{S}^{(k)}(V)$ and $\Lambda^{(k)}(V)$ respectively. These also form graded vector spaces.

The two graded vector spaces $\mathcal{S}(V)$ and $\Lambda(V[-1])$ are isomorphic as vector spaces (forgetting the grading). Defining dec_n by

$$\mathrm{dec}_n(x_1 \otimes \cdots \otimes x_k) := (-1)^{\sum_{i=1}^{k}(k-i)|x_i|} x_1 \otimes \cdots \otimes x_k$$

yields an isomorphism between $\mathcal{S}^{(k)}(V)[-k]$ and $\Lambda^{(k)}(V[-1])$ which is known as the *décalage-isomorphism.*

The graded symmetric algebra $\mathcal{S}(V)$ of a graded vector space V inherits the structure of a coassociative coalgebra from $(T(V), \Delta)$, see [**Sta1**] for instance: the restriction of the coproduct Δ to $\mathcal{S}(V)$ has image in $\mathcal{S}(V) \otimes \mathcal{S}(V)$. Moreover it is graded cocommutative, i.e. its image is again invariant under the even action of the permutation groups.

DEFINITION 1.9. Let V be a graded vector space. Given a family of morphisms

$$(m_k : \mathcal{S}^{(k)}(V) \to V[1])_{k \in \mathbb{N}}$$

of graded vector spaces, the *associated family of Jacobiators* is the family of morphisms

$$(J_k : \mathcal{S}^{(k)}(V) \to V[2])_{k \in \mathbb{N}}$$

given by

$$\begin{aligned} J_k(x_1 \otimes \cdots \otimes x_k) &:= \\ &= \sum_{r+s=k} \sum_{\sigma \in (r,s)-\text{shuffles}} \mathrm{sign}(\sigma) m_{(s+1)}(m_r(x_{\sigma_1} \otimes \cdots \otimes x_r) \otimes x_{(r+1)} \otimes \cdots \otimes x_k) \end{aligned}$$

on the tensor product of homogeneous elements $x_1, \ldots, x_n \in V$. Here (r,s)-shuffles are all permutations σ of a set with k elements such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(k)$ holds and $\mathrm{sign}(\sigma)$ is the sign induced by the even representation of Σ_k on $\mathcal{T}^{(k)}(V)$.

A family of morphisms

$$(m_k : \mathcal{S}^{(k)}(V) \to V[1])_{k \in \mathbb{N}}$$

defines an $L_\infty[1]$-*algebra* on the graded vector space V if the associated family of Jacobiators vanishes.

An L_∞-*algebra* on V is an $L_\infty[1]$-algebra on $V[1]$.

REMARK 1.10. Observe that our definition of $L_\infty[1]$-algebra is slightly non-standard because it includes a possible non-vanishing $\mathbb{R}$-linear map $m_0 : \mathbb{R} \to V_2[2]$ which can be interpreted as an element of V_2. We refer to L_∞-algebras with vanishing zero order term as *flat*.

It is straightforward to check that the notion of L_∞-algebra from above specializes to the notion of a differential graded Lie algebra if all structure maps m_n except m_1 and m_2 vanish.

DEFINITION 1.11. A triple $(V, d, [\cdot,\cdot])$ is a *differential graded Lie algebra* (over $\mathbb{R}$) if

(a) V is a graded vector space,
(b) d is a morphism from V to $V[1]$,
(c) $[\cdot,\cdot]$ is a morphism from $V \otimes V$ to V

such that the following conditions are fulfilled:

(a') d is a coboundary operator, i.e. $d \circ d = 0$,
(b') $[\cdot,\cdot]$ is a graded Lie bracket, i.e. $[\cdot,\cdot]$ is graded skew-symmetric –
$$[x, y] = -(-1)^{|x||y|}[y, x]$$
– and the graded Jacobi identity
$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$$
is satisfied for all homogeneous x, y and z in V and
(c') the graded Leibniz-identity
$$d([x, y]) = [dx, y] + (-1)^{|x|}[x, dy]$$
holds for all homogeneous x and y in V.

DEFINITION 1.12. Let V and W be two graded vector spaces equipped with $L_\infty[1]$-algebra structures $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ respectively. A *morphism* from $(V, (m_k)_{k\in\mathbb{N}})$ to $(W, (m_k)_{k\in\mathbb{N}})$ is a family of morphisms
$$F := (f_k : \mathcal{S}^{(k)}(V) \to W)_{k\in\mathbb{N}}$$
such that the two families of morphisms $\mathcal{S}^{(k)}(V) \to W[1]$ given by
$$\sum_{r+s=k} \sum_{\sigma\in(r,s)-\text{shuffles}} \text{sign}(\sigma) f_{(s+1)}(m_r(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}) \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(n)}) \quad \text{and}$$
$$\sum_{l=1}^{n} \sum_{j_1+\cdots+j_l=k} \sum_{\tau\in\Sigma_k} \frac{sign(\tau)}{l! j_1! \cdots j_l!} n_l(f_{j_1}(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma_{j_1}}) \otimes \cdots$$
$$\cdots \otimes f_{j_l}(x_{\sigma(k-(j_1+\cdots+j_{(l-1)}))} \otimes \cdots \otimes x_{\sigma(n)}))$$
on homogeneous elements $x_1, \ldots, x_k \in V$ coincide.

REMARK 1.13. The definition of an $L_\infty[1]$-morphism given in Definition 1.12 is rather cumbersome to work with. For instance it is not evident from the Definition that the classes of L_∞-algebras and morphisms from a category. A conceptual approach to $L_\infty[1]$-algebras and their morphisms was given by Stasheff in [**Sta1**]. As mentioned in Remark 1.8, the symmetric algebra $\mathcal{S}(V)$ of a graded vector space V inherits the structure of a coalgebra from $(\mathcal{T}(V), \Delta)$. Stasheff noticed that every family of maps

$$(m_k : \mathcal{S}^{(k)}(V) \to V[1])_{k \in \mathbb{N}}$$

yields a coderivation M of $(\mathcal{S}(V), \Delta)$ of degree $+1$, i.e. an endomorphism M of the graded vector space $\mathcal{S}(V)$ such that

$$\Delta \circ M = (M \otimes id + id \otimes M) \circ \Delta$$

holds. Here $\otimes$ is the tensor product which takes the even representation on $\mathcal{T}(V)$ into account, i.e. $(id \otimes M)(x \otimes y) := (-1)^{|x|} x \otimes M(y)$ for all homogeneous x and y in $\mathcal{S}(V)$.

The formula for M in terms of $(m_k)_{k \in \mathbb{N}}$ is

$$\begin{aligned} &M(x_1 \otimes \cdots \otimes x_k) := \\ &\quad = \sum_{r+s=k} \sum_{\sigma \in (r,s)-\text{shuffles}} \operatorname{sign}(\sigma) m_r(x_{\sigma_1} \otimes \cdots x_{\sigma(r)}) \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(k)} \end{aligned}$$

where $x_1, \ldots, x_k$ are homogeneous elements of V. It is straightforward to check that M is a coderivation.

On the other hand any coderivation M yields a family of maps $(m_k)_{k \in \mathbb{N}}$. Just consider the composition

$$\mathcal{S}^k(V) \hookrightarrow \mathcal{S}(V) \xrightarrow{M} \mathcal{S}(V)[1] \to V[1]$$

where the last map denotes the first suspension of the projection $\mathcal{S}(V) \to V$. One can check that the association between family of maps $(m_k)_{k \in \mathbb{N}}$ and coderivations M is a bijection, i.e. there is only one way to extend a family of structure maps to a coderivation and it is given by the above formula.

In Definition 1.9 a family of Jacobiators $(J_k)_{k \in \mathbb{N}}$ was associated to any family of maps $(m_k)_{k \in \mathbb{N}}$. Stasheff observed that the family of Jacobiators can also be interpreted in terms of coderivations. The vector space of coderivations of the coalgebra $(\mathcal{S}(V), \Delta)$ of some fixed degree comes along with the structure of a graded Lie algebra: let Q_1 and Q_2 be two coderivation of degree $|Q_1|$ and $|Q_2|$ respectively. We define

$$[Q_1, Q_2] := Q_1 \circ Q_2 - (-1)^{|Q_1||Q_2|} Q_2 \circ Q_1$$

which can be easily checked to be a coderivation of degree $|Q_1| + |Q_2|$. Consider the coderivation M of degree 1 corresponding the a family of morphisms $(m_k : \mathcal{S}^{(k)}(V) \to V[1])_{k \in \mathbb{N}}$. Then $1/2[M, M]$ is a coderivation of degree 2 which corresponds to the family of Jacobiators associated to $(m_k)_{k \in \mathbb{N}}$.

Consequently there is a one-to-one correspondence between $L_\infty[1]$-algebra structures on a graded vector space V and degree 1 coderivations of the coalgebra $(\mathcal{S}(V),\Delta)$ that satisfy the equation $[M,M]=0$. Because the degree of M is odd, we obtain

$$[M,M]=2(M\circ M),$$

i.e. $[M,M]$ vanishes if and only if $M\circ M$ does. A coderivation M of degree 1 that satisfies $M\circ M=0$ is called a *codifferential*. Hence there is a one-to-one correspondence between $L_\infty[1]$-algebra structures on a graded vector space V and codifferentials on $(\mathcal{S}(V),\Delta)$.

Now suppose F is a morphism of coalgebras from $(\mathcal{S}(V),\Delta)$ to $(\mathcal{S}(W),\Delta)$, i.e.

$$F\otimes F\circ\Delta=\Delta\circ F$$

holds. Any such morphism yields a family of morphisms $(f_k)_{k\in\mathbb{N}}$ of graded algebras

$$\mathcal{S}^{(k)}(V)\hookrightarrow\mathcal{S}(V)\xrightarrow{F}\mathcal{S}(W)\to W.$$

On the other hand any family of morphisms

$$(f_k:\mathcal{S}^{(k)}(V)\to W)_{k\in\mathbb{N}}$$

yields a morphism of coalgebras F from $(\mathcal{S}(V),\Delta)$ to $(\mathcal{S}(W),\Delta)$. The formula for F in terms of $(f_k)_{k\in\mathbb{N}}$ is

$$\begin{aligned}F(x_1\otimes\cdots\otimes x_k):=\sum_{l=1}^{n}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{sign(\tau)}{l!j_1!\cdots j_l!}f_{j_1}(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma_{j_1}})\otimes\cdots\\ \cdots\otimes f_{j_l}(x_{\sigma(j_1+\cdots+j_{(l-1)}+1)}\otimes\cdots\otimes x_{\sigma(n)})\end{aligned}$$

where $x_1,\dots,x_k$ are homogeneous elements of V. Given any family of maps $(f_k)_{k\in\mathbb{N}}$ as above the formula for F just given is the unique extension to a morphism of coalgebras. Hence there is a one-to-one correspondence between families of morphisms

$$(f_k:\mathcal{S}^{(k)}(V)\to W)_{k\in\mathbb{N}}$$

and morphisms of coalgebras from $(\mathcal{S}(V),\Delta)$ to $(\mathcal{S}(W),\Delta)$.

Let V and W be two graded vector spaces equipped with the structure of $L_\infty[1]$-algebras. We saw that this is equivalent to coderivations M and N of the coalgebras $(\mathcal{S}(V),\Delta)$ and $(\mathcal{S}(W),\Delta)$ respectively. We denote the corresponding family of morphisms by $(m_k:\mathcal{S}^{(k)}(V)\to V[1])_{k\in\mathbb{N}}$ and $(n_k:\mathcal{S}^{(k)}(W)\to W[1])_{k\in\mathbb{N}}$ respectively. Given a morphism of coalgebras $F:(\mathcal{S}(V),\Delta)\to(\mathcal{S}(W),\Delta)$ both $F\circ Q_1$ and $Q_2\circ F$ yield families of maps $(a_k:\mathcal{S}^{(k)}(V)\to W[1])_{k\in\mathbb{N}}$ and $(b_k:\mathcal{S}^{(k)}(V)\to W[1])_{k\in\mathbb{N}}$ which are given by

$$\begin{aligned}&\mathcal{S}^{(k)}(V)\hookrightarrow\mathcal{S}(V)\xrightarrow{F\circ Q_1}\mathcal{S}(W)[1]\xrightarrow{\mathrm{pr}}W[1]\quad\text{and}\\ &\mathcal{S}^{(k)}(V)\hookrightarrow\mathcal{S}(V)\xrightarrow{Q_2\circ F}\mathcal{S}(W)[1]\xrightarrow{\mathrm{pr}}W[1]\quad\text{respectively.}\end{aligned}$$

The left-hand side of the defining identity for $(f_k)_{k\in\mathbb{N}}$ to be an $L_\infty[1]$-morphism from $(V,(m_k)_{k\in\mathbb{N}})$ to $(W,(n_k)_{k\in\mathbb{N}})$ in Definition 1.12 is exactly a_k, while the right-hand side is exactly b_k.

If we assume that $F:\mathcal{S}(V)\to\mathcal{S}(W)$ is not only a morphism of coalgebras but a morphism of complexes, i.e. $F\circ Q_1=Q_2\circ F$ hold, then $(a_k)_{k\in\mathbb{N}}=(b_k)_{k\in\mathbb{N}}$ and hence the family of morphisms $(f_k:\mathcal{S}^{(k)}(V)\to W[1])_{k\in\mathbb{N}}$ corresponding to F defines a morphism of $L_\infty[1]$-algebras from $(V,(m_k)_{k\in\mathbb{N}})$ to $(W,(n_k)_{k\in\mathbb{N}})$. On the other hand suppose $(f_k:\mathcal{S}^{(k)}(V)\to W)_{k\in\mathbb{N}}$ is an $L_\infty[1]$-morphism from $(V,(m_k)_{k\in\mathbb{N}})$ to $(W,(n_k)_{k\in\mathbb{N}})$. The family of maps $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are equal, i.e. $\mathrm{pr}\circ F\circ Q_1=\mathrm{pr}\circ Q_2\circ F$ holds. It can be check that this implies $F\circ Q_1=Q_2\circ F$, i.e. $F:(\mathcal{S}(V),\Delta)\to(\mathcal{S}(W),\Delta)$ is a morphism of complexes. Summing up, there is a one-to-one correspondence between $L_\infty[1]$-morphisms and morphisms of coalgebras that are additionally morphisms of complexes.

We equip the class of $L_\infty[1]$-algebras with the composition inherited from the one-to-one correspondence with certain morphisms of coalgebras and complexes. In particular an $L_\infty[1]$-*isomorphism* is an $L_\infty[1]$-morphism with an inverse.

Suppose V and W are graded vector spaces equipped with the structures of flat $L_\infty[1]$-algebras, i.e. $(V,(m_k)_{k\in\mathbb{N}})$ and $(W,(n_k)_{k\in\mathbb{N}})$ are $L_\infty[1]$-algebras such that $m_0:\mathbb{R}\to V_2[2]$ and $n_0:\mathbb{R}\to W_2[2]$ vanish identically. The structure maps $m_1:V\to V[1]$ and $n_1:W\to W[1]$ both square to zero, hence they are coboundary operators on V and W. We denote the associated cohomologies by $H(V,m_1)$ and $H(W,n_1)$ respectively. Assume the family of maps $(f_k:\mathcal{S}^{(k)}(V)\to W)_{k\in\mathbb{N}}$ defines an $L_\infty[1]$-morphism between $(V,(m_k)_{k\in\mathbb{N}})$ and $(W,(n_k)_{k\in\mathbb{N}})$. This implies that $f_1:(V,m_1)\to(W,n_1)$ is not only a morphism of graded vector spaces but of complexes. In particular it induces a morphism $[f_1]:H(V,m_1)\to H(W,m_2)$ between the cohomologies. If this morphism is an isomorphism of graded vector spaces the $L_\infty[1]$-morphism $(f_k:\mathcal{S}^{(k)}(V)\to W)_{k\in\mathbb{N}}$ is called an $L_\infty[1]$ *quasi-isomorphism*. This special class of morphisms plays an important role in the homotopy theory of L_∞-algebras.

The notion of L_∞*-morphism, -isomorphism* and *quasi-isomorphism* is the translation of the corresponding notion for $L_\infty[1]$-algebras with the help of the décalage-isomorphism, see Remark 1.8.

Definition 1.14. Let $(V,(m_k)_{k\in\mathbb{N}})$ be an L_∞-algebra. A *Maurer–Cartan element* of $(V,(m_k)_{k\in\mathbb{N}})$ is an element $x\in V_1$ such that

$$\sum_{k\geq 0}\frac{1}{k!}m_k(x\otimes\cdots\otimes x)=0\quad\text{holds.}$$

Remark 1.15. Observe that our notion of Maurer–Cartan elements is not complete: the statement requires a notion of convergence of the series

$$\Big(\sum_{k=0}^{N}\frac{1}{k!}m_k(x\otimes\cdots\otimes x)\Big)_{N\in\mathbb{N}}.$$

One way around this lack of information is to introduce a formal parameter λ and to work with the λ-linear extension of the L_∞-algebra $(V,(m_k)_{k\in\mathbb{N}})$ to $V[[\lambda]]$. If one only consideres elements of $\lambda V_1[[\lambda]]$ the series $\sum_{k\geq 0}\frac{1}{k!}m_k(x\otimes\cdots\otimes x)$ automatically converges in the λ-adic topology. We refer to Maurer–Cartan elements in $\lambda V_1[[\lambda]]$ as *formal Maurer–Cartan elements* of $(V,(m_k)_{k\in\mathbb{N}})$.

However we will be only interested in Maurer–Cartan elements of L_∞-algebras whose family of structure maps $(m_k)_{k\in\mathbb{N}}$ is bounded, i.e. $m_N = 0$ for N sufficiently large. In fact we are interested primarily in the case where all structure maps m_k except for $k = 1$ and 2 vanish, i.e. the case of differential graded Lie algebras. Then the series $(\sum_{k\geq 0}\frac{1}{k!}m_k(x\otimes\cdots\otimes x))$ reduces to a finite sum and no convergence issues arise.

It is well-known that every L_∞-morphism from $(V,(m_k)_{k\in\mathbb{N}})$ to $(W,(n_k)_{k\in\mathbb{N}})$ induces a map from the set of *formal* Maurer–Cartan elements of $(V,(m_k)_{k\in\mathbb{N}})$ to the set of formal Maurer–Cartan elements of $(W,(n_k)_{k\in\mathbb{N}})$. It can be shown that L_∞ quasi-isomorphisms induce bijections between the sets of equivalence classes of formal Maurer–Cartan elements.

2. Homological Transfer

Definition 2.1. Let (X,d) be a chain complex whose cohomology $H(X,d)$ we denote by H. We interpret H as a complex with vanishing coboundary operator.

A set of *contraction data* for (X,d) is a triple of morphisms (i,p,h) of graded vector spaces where

(i) $i : (H,0) \to (V,d)$ is an injective morphism of chain complexes,
(ii) $p : (V,d) \to (H,0)$ is a surjective morphism of chain complexes and
(iii) h is a morphism from V to $V[-1]$

such that

(i') $p\circ i = \mathrm{id}_H$,
(ii') $[h,d] := h\circ d + d\circ h = \mathrm{id} - i\circ p$ and
(iii') $h\circ h = 0$, $h\circ i = 0$ and $p\circ h = 0$

hold.

We sum up this situation diagrammatically as

$$(H,0) \underset{p}{\overset{i}{\rightleftarrows}} (X,d), h.$$

Theorem 2.2. *Let (X,d) be a chain complex and (i,p,h) contraction data for (X,d). Furthermore we assume that X is equipped with the following structures:*

1.) *there is a finite filtration of X, i.e. a collection of graded vector subspaces*

$$X = \mathcal{F}_0(X) \supseteq \mathcal{F}_1(X) \supseteq \cdots \supseteq \mathcal{F}_k(X) \supseteq \mathcal{F}_{(k+1)}(X) \supseteq \cdots$$

such that the following conditions are satisfied:

- $\mathcal{F}_N(X) = \{0\}$ *for* N *sufficiently large,*
- $d(\mathcal{F}_k(X)) \subseteq \mathcal{F}_k(X)$ *for all* $k \geq 0$ *and*
- $h(\mathcal{F}_k(X)) \subseteq \mathcal{F}_k(X)$ *for all* $k \geq 0$.

2.) $(X, D, [\cdot,\cdot])$ *is a differential graded Lie algebra such that*

$$(D-d)(\mathcal{F}_k(X)) \subset \mathcal{F}_{(k+1)}(X) \text{ holds for all } k \geq 0.$$

Then the cohomology H *of* (X, d) *inherits an* induced L_∞-algebra structure *which comes along with a well-defined* L_∞*-morphism to* $(X, D, [\cdot,\cdot])$.

Remark 2.3. Theorem 2.2 is one of the central tools throughout Chapter 4. Since we are not primarily interested in this transfer-procedure of L_∞-algebras along contraction data for its own sake but rather as a tool, we did not state Theorem 2.2 in the largest possible generality. In particular, one can check that the L_∞-morphism from H equipped with the induced L_∞-algebra structure to $(X, D, [\cdot,\cdot])$ is actually an L_∞ quasi-isomorphism. But this is easy to check by hand in all the cases of interest to us.

The conceptual proof of Theorem 2.2 is straightforward and can be found in [**GL**] for instance. One uses the interpretation of the differential graded Lie algebra structure on X as a codifferential on $\mathcal{S}(X[1])$, see Remark 1.13. Explicit transfer formulae for this codifferential exist and one obtains a codifferential on $\mathcal{S}(H[1])$. This in turn yields the induced L_∞-algebra structure on H. Additionally one obtains a morphism of coalgebras and complexes from $\mathcal{S}(H[1])$ to $\mathcal{S}(X[1])$. By Remark 1.13 such a morphism is equivalent to a morphism of L_∞-algebras from H to X.

Although Theorem 2.2 establishes the existence of a transfer-procedure along contraction data, we need a more concrete description of the induced L_∞-algebra and of the L_∞ quasi-isomorphism between H and X. Such a description was first given in the setting of A_∞-algebras: in [**Me**] inductive formulae were presented for the structure maps of the induced structure and in [**KSo**] an interpretation in terms of certain graphs was provided. Similar descriptions are known to exist for the transfer of L_∞-algebras as well, but we need a slight generalization of the setting presented in [**Me**] and [**KSo**] since we allow the coboundary operator D to deviate from d.

Remark 2.4. For the moment we assume that the graded Lie bracket $[\cdot,\cdot]$ on X vanishes. In that case we only have to care about the coboundary operator D on X and the induced L_∞-algebra structure will reduce to a coboundary operator on $H = H(X, d)$.

Lemma 2.5. *Let* (X, d) *be a complex equipped with contraction data* (i, p, h) *and a filtration*

$$X = \mathcal{F}_0(X) \supseteq \mathcal{F}_1(X) \supseteq \cdots \supseteq \mathcal{F}_k(X) \supseteq \mathcal{F}_{(k+1)}(X) \supseteq \cdots$$

such that the following conditions are satisfied:

- *$\mathcal{F}_N(X) = \{0\}$ for N sufficiently large,*
- *$d(\mathcal{F}_k(X)) \subseteq \mathcal{F}_k(X)$ for all $k \geq 0$ and*
- *$h(\mathcal{F}_k(X)) \subseteq \mathcal{F}_k(X)$ for all $k \geq 0$.*

Moreover (X, D) is a complex such that

$$(D - d)(\mathcal{F}_k(X)) \subset \mathcal{F}_{(k+1)}(X) \text{ holds for all } k \geq 0.$$

Set $D_R := D - d$. The formula

$$p \circ D_R \circ \left(\sum_{k \geq 0} (-hD_R)^k \right)$$

defines a coboundary operator $\mathcal{D}$ on H. Furthermore

$$\left(\sum_{k \geq 0} (-hD_R)^k \right) \circ i$$

defines a morphism of complexes $\tilde{i}$ from $(H, \mathcal{D})$ to (X, D).

Proof. Observe that the term $(-hD_R)^k$ maps $\mathcal{F}_l(X)$ to $\mathcal{F}_{(k+l)}(X)$. Since the filtration $\mathcal{F}_\bullet(X)$ is bounded from above, the series that define $\mathcal{D}$ and $\tilde{i}$ respectively are finite sums.

The identities $D^2 = (d + D_R)^2 = 0$ and $d^2 = 0$ imply

$$D_R \circ d + d \circ D_R + D_R \circ D_R = 0.$$

We set $\tilde{\mathcal{D}} := D_R \circ \sum_{k \geq 0} (-hD_R)^k$ and compute

$$\begin{aligned}
-d\tilde{\mathcal{D}} &= (-dD_R) \left(\sum_{k \geq 0} (-hD_R)^k \right) \\
&= (D_R D_R) \left(\sum_{k \geq 0} (-hD_R)^k \right) + (D_R d) \left(\sum_{k \geq 0} (-hD_R)^k \right) \\
&= D_R \tilde{\mathcal{D}} + (D_R d) - (D_R dh) \tilde{\mathcal{D}} \\
&= D_R d + D_R ip \tilde{\mathcal{D}} + D_R hd \tilde{\mathcal{D}}.
\end{aligned}$$

Applying this formula iteratively and observing that $D_R hd$ increases the filtration degree by 1 yields

$$-d\tilde{\mathcal{D}} = \tilde{\mathcal{D}} d + \tilde{\mathcal{D}} ip \tilde{\mathcal{D}}$$

and consequently

$$\mathcal{D}^2 = p(\tilde{\mathcal{D}} ip \tilde{\mathcal{D}}) i = p(-d\tilde{\mathcal{D}} - \tilde{\mathcal{D}} d) i = 0.$$

Finally we rewrite $\tilde{i}$ as $(\mathrm{id} - h\tilde{\mathcal{D}})i$ and calculate

$$\begin{aligned} D \circ \tilde{i} &= (d + D_R)(\mathrm{id} - h\tilde{\mathcal{D}})i = d(-h\tilde{\mathcal{D}})i + \mathcal{D}i \\ &= ip\tilde{\mathcal{D}}i + hd\tilde{\mathcal{D}}i = ip\tilde{\mathcal{D}}i - h\tilde{\mathcal{D}}ip\tilde{\mathcal{D}} \\ &= (id - h\tilde{\mathcal{D}})i \circ p\tilde{\mathcal{D}}i = \tilde{i} \circ \mathcal{D}, \end{aligned}$$

i.e. $\tilde{i}$ is a chain map from $(H, \mathcal{D})$ to (X, D). □

REMARK 2.6. In the special case treated in Lemma 2.5 the induced L_∞-algebra structure on $H(X, d)$ was given as the sum of terms – $D_R \circ (-hD_R)^k$ – associated to every integer $k \in \mathbb{N}$. It turns out that the right generalization to the setting of Theorem 2.2 is given by associating morphisms to certain graphs.

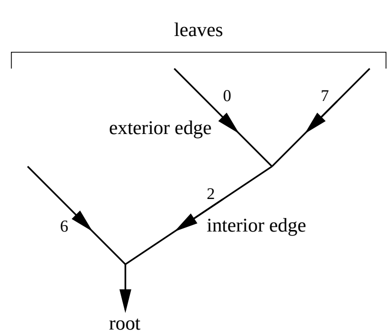
A *decorated oriented trivalent tree* T is a finite connected graph without any loops that consists of

(i) *interior vertices* that are all trivalent,
(ii) *exterior vertices* of valency 1,
(iii) oriented edges

such that

(i') at every interior vertex v two edges are oriented towards v and one is oriented away from v and
(ii') the set of exterior vertices is a disjoint union of the set consisting of vertices such that the connected edges point away from them – we call them *leaves* – and a unique vertex of valency 1 with an edge oriented towards it – we call it the *root*.

An *orientation* of such a tree is given by an order of all incoming edges at the interior vertices, i.e. for every interior vertex, one of the incoming edges is the "left" one and the other one is the "right" one. The *decoration* is an assignment of a natural number to every edge of the tree. We will sometimes refer to edges not connected to any leaf or to the root as *interior edges* and to all others as *exterior edges*. Moreover we require that the edge of the graph that only consists of a single leaf which is connected directly to the root to be decorated by a positive natural number.



We denote the set of all such decorated oriented trivalent trees by $\mathbb{T}$. Clearly we have a decomposition

$$\mathbb{T} = \bigsqcup_{n \geq 1} \mathbb{T}(n)$$

where $\mathbb{T}(n)$ is the set of decorated oriented trivalent trees with exactly n leaves.

Forgetting the orientation data assigned to the interior vertices yields decorated *unoriented* trivalent trees. We denote the set of decorated unoriented trivalent trees by $[\mathbb{T}]$. There is a natural projection

$$[\cdot] : \mathbb{T} \to [\mathbb{T}]$$

that respect the decomposition of $\mathbb{T}$ and that of $[\mathbb{T}]$:

$$[\mathbb{T}] = \bigsqcup_{n \geq 1} [\mathbb{T}](n) = \bigsqcup_{n \geq 1} [\mathbb{T}(n)].$$

We define $|\operatorname{Aut}(T)|$ for T a decorated oriented trivalent tree to be the cardinality of the group of automorphisms of the underlying decorated unoriented tree $[T]$.

Observe that the set of leaves of a decorated oriented trivalent tree is ordered thanks to the orientation: consider two leaves L and L'. There is a unique oriented path from each of these leaves to the root. These paths will meet at some vertex v for the first time. If the edge e_L that lies on the path from L to the root and points towards v is left from the edge $e_{L'}$ that lies on the path from L' to the root and points towards v we say that L is *left* from L'. This relation equips the set of leaves with an total ordering and we number the leaves corresponding to this order.

REMARK 2.7. Consider the complex (X, d) equipped with all the structures of Theorem 2.2, i.e. contraction data (i, p, h), a compatible finite filtration $\mathcal{F}_\bullet(X)$ and the structure of a differential graded Lie algebra $(X, D, [\cdot,\cdot])$. Differential graded Lie algebras are special cases of L_∞-algebras, so $X[1]$ comes along with the structure of an $L_\infty[1]$-algebra. The differential is unchanged but the bracket $[\cdot,\cdot]$ picks up additional signs from the décalage-isomorphism, see Remark 1.8. We denote the shifted bracket by $\{\cdot,\cdot\}$ from now on, i.e. $(X[1], D, \{\cdot,\cdot\})$ is an $L_\infty[1]$-algebra.

To any decorated oriented trivalent tree T with k leaves we associated a map

$$\hat{m}(T) : (X[1])^{\otimes k} \to X[2]$$

by the following procedure: first embed the tree into the plane in a way compatible with its orientation. Put a $\{\cdot,\cdot\}$ at every trivalent vertex and l copies of $D_R = D - d$ on every edge decorated by l. Between any two such consecutive operations place $-h$ along the tree. Compose all these maps in the order given by the orientation of the edges. Given $x_1 \otimes \cdots \otimes x_k \in (X[1])^{\otimes k}$ place x_n at the n'th vertex and compute the image under the composed structure maps.

It is easy to check that the map

$$\hat{M}(T) := \sum_{\sigma\in\Sigma_k} \frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{m}(T)$$

does not depend on the orientation of T. Here σ^* is the action of $\sigma \in \Sigma_k$ on $(X[1])^{\otimes k} \to X[2]$ induced from the even representation of Σ_k on $(X[1])^{\otimes k}$, see Remark 1.6. Hence we get a map well-defined map

$$\hat{M}([T]) : \mathcal{S}^{(k)}(X[1]) \to X[2], \quad \hat{M}([T]) := \sum_{\sigma\in\Sigma_k} \frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{m}(T).$$

Finally we set

$$\hat{M}_k : \mathcal{S}^{(k)}(X[1]) \to X[2], \quad \hat{M}_k := \sum_{[T]\in[\mathbb{T}](k)} \hat{M}([T])$$

and $M_k := p \circ \hat{M}_k \circ i^{\otimes k}$.

Proposition 2.8. *Let (X, d) be a complex equipped with all the structures of Theorem 2.2, i.e. contraction data (i, p, h), a compatible finite filtration $\mathcal{F}_\bullet(X)$ and the structure of a differential graded Lie algebra $(X, D, [\cdot,\cdot])$.*

Then $(H[1], (M_k : \mathcal{S}^{(k)}(H[1]) \to H[2])_{k\in\mathbb{N}})$ is an $L_\infty[1]$-algebra where $(M_k)_{k\in\mathbb{N}}$ is the family of structure maps introduced in Remark 2.7. We call the corresponding L_∞-algebra structure on H the induced L_∞-algebra.

Proof. First observe that if we fix the number of leaves there are only finitely many decorated oriented trivalent trees T with that number of leaves for which $\hat{m}(T)$ is non-zero because if a edge is decorated by l the filtration degree increases by l and the filtration degree is bounded from above. Hence M_k is well-defined.

We have to check that the family of Jacobiators $(J_k)_{k\in\mathbb{N}}$ associated to $(M_k)_{k\in\mathbb{N}}$ vanishes. We define a family of maps $(\hat{J}_k : \mathcal{S}^{(k)}(X[1]) \to X[3])_{k\in\mathbb{N}}$ by

$$\begin{aligned}\hat{J}_k(x_1\otimes\cdots\otimes x_k) &:= \\ &= \sum_{r+s=k}\ \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma)\hat{M}_{(s+1)}(ip\hat{M}_r(x_{\sigma_1}\otimes\cdots\otimes x_r)\otimes x_{(r+1)}\otimes\cdots\otimes x_k).\end{aligned}$$

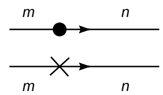
Observe that $J_k = p\hat{J}_k i^{\otimes k}$ holds.

We claim that

$$-d\hat{M}_k \circ i^{\otimes k} = \hat{J}_k \circ i^{\otimes k}$$

is true. This would immediately imply that the family of Jacobiators $(J_k)_{k\in\mathbb{N}}$ vanishes.

In order to prove this claim we extend the set of decorations for the trees we consider: we allow to add one special edge which is either marked by a "$\bullet$" or a "$\times$" and require that this special edge is decorated by two natural numbers:



We call decorated oriented trivalent trees with a special edge of the first kind *pointed* and with a special edge of the second kind *truncated*. We denote the set of pointed decorated oriented trivalent trees by $\mathbb{T}^{\bullet}$ and the set of truncated decorated oriented trivalent trees by $\mathbb{T}^{\times}$ respectively.

Moreover we extend the map $\hat{m}$ that associates to any decorated oriented trivalent tree with k leaves a morphism $(X[1])^{\otimes k} \to X[2]$ to the set of pointed and truncated decorated oriented trivalent trees. So let T be a pointed or truncated decorated oriented trivalent tree. The first step of the procedure to build $\hat{m}(T)$ is the same as if we would replace the special edge decorated by (m,n) be an ordinary edge decorated by $m+n$. Then remove the $-h$ placed after m copies of D_R at this edge and replace it by ip in case the special edge was a pointed one and by id in case the special edge was a truncated one. Finally one adds the sign given by (-1) to the sum of the degrees of all inputs left to the special edge.

It is straightforward to check that

$$
\begin{aligned}
&\hat{J}_k(x_1 \otimes \cdots \otimes x_k) = \\
&\quad = \sum_{\sigma\in\Sigma_k} \sum_{r+s=k} \sum_{[S]\in[[\mathbb{T}]](s+1)} \sum_{[R]\in[[\mathbb{T}]](r)} \frac{\operatorname{sign}(\sigma)}{|\operatorname{Aut}(S)||\operatorname{Aut}(R)|} \cdots \\
&\qquad\qquad \hat{m}(S)\left(\left(ip\hat{m}(R)(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)})\right) \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(k)}\right) \\
&\quad = \sum_{\sigma\in\Sigma_k} \sum_{[T]\in[\mathbb{T}](k)} \frac{\operatorname{sign}(\sigma)}{|\operatorname{Aut}(T)|} m(P(T))(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)})
\end{aligned}
$$

holds. Here $P(T)$ is the sum of all ways to turn an ordinary decorated edge of T into a pointed decorated edge.

Consequently the claimed equality $-d\hat{M}_k i^{\otimes k} = \hat{J}_k i^{\otimes k}$ is equivalent to

$$
-d\hat{M}_k i^{\otimes k} = \left(\sum_{\sigma\in\Sigma_k} \sum_{[T]\in[\mathbb{T}](k)} \frac{1}{|\operatorname{Aut}(T)|} \sigma^* \hat{m}(P(T)) \right) i^{\otimes k}.
$$

We prove the latter identity by induction over the number of leaves. For $k = 1$ the identity simply reduces to $-d\tilde{\mathcal{D}}i = \tilde{\mathcal{D}}ip\tilde{\mathcal{D}}i$ where $\tilde{\mathcal{D}} = D_R \sum_{n\geq 0}(-hD_R)^n$, see the proof of Lemma 2.5.

The inductive step uses the identities

$$
-d\hat{m}(\xrightarrow{n}) = \sum_{r+s=n+1} \hat{m}(\overset{r}{\times}\!\xrightarrow{s}) - \sum_{r+s=n} \hat{m}(\overset{r}{\times}\!\xrightarrow{s}) + \sum_{r+s=n} \hat{m}(\overset{r}{\bullet}\!\xrightarrow{s}) + \hat{m}(\xrightarrow{n})d
$$

and

$$
\begin{aligned}
-d\{X,Y\} &= \{dX,Y\} + (-1)^{|X|}\{X,dY\} + \\
&+ \{D_R X, Y\} + (-1)^{|X|}\{X, D_R Y\} + D_R\{X,Y\}.
\end{aligned}
$$

Iterative use of these identities in the computation of

$$-dM_k i^{\otimes k} = -d\left(\sum_{\sigma\in\Sigma_k}\sum_{[T]\in[\mathbb{T}](k)}\frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{m}(T)\right)i^{\otimes k}$$

leads to

$$\left(\sum_{\sigma\in\Sigma_k}\sum_{[T]\in[\mathbb{T}](k)}\frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{m}(P(T))\right)i^{\otimes k}+\left(\sum_{\sigma\in\Sigma_k}\sum_{[T]\in[\mathbb{T}](k)}\frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{m}(X(T))\right)i^{\otimes k}$$

where $X(T)$ is the sum of all ways to change an ordinary edge of T that is decorated by 0 into a truncated one decorated by $(0,0)$. The second sum contains terms of the form

$$\begin{aligned}&x_1\otimes\cdot\otimes x_l\mapsto\\&\quad\sum_{\sigma\in\Sigma_l}\sum_{r+t+s=l}\frac{1}{2}\sum_{[U]\in[\mathbb{T}](r),[V]\in[\mathbb{T}](s),[W]\in[\mathbb{T}](t)}\Big(\{(\{-h\hat{m}(U)(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}),\cdots\\&\qquad\cdots,-h\hat{m}(V)(x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(r+s)})\}),-h\hat{m}(W)(x_{\sigma(r+s+1)}\otimes\cdots\otimes x_{\sigma(l)}\Big).\end{aligned}$$

Since the expression in the last two lines is of the form $\{\{a,b\},c\}$ and the sum runs over all permutations with appropriate signs, the total sum vanishes thanks to the graded Jacobi identity for $\{\cdot,\cdot\}$. Consequently

$$-d\hat{M}_k i^{\otimes i} = \left(\sum_{\sigma\in\Sigma_k}\sum_{[T]\in[\mathbb{T}](k)}\frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{m}(P(T))\right)i^{\otimes k} = \hat{J}_k i^{\otimes i}$$

and $J_k := p\hat{J}_k i^{\otimes k}$ vanishes. □

REMARK 2.9. We start with the same input data as in Remark 2.7: (X,d) is a complex equipped with contraction data (i,p,h), a compatible finite filtration $\mathcal{F}_\bullet(X)$ and the structure of a differential graded Lie algebra $(X,D,[\cdot,\cdot])$ such that all conditions stated in Theorem 2.2 are satisfied. Again we translate the graded Lie bracket $[\cdot,\cdot]$ into a graded symmetric operation $\{\cdot,\cdot\}$ on $X[1]$.

To any decorated oriented trivalent tree T with k leaves we associated a map

$$\hat{n}(T):(X[1])^{\otimes k}\to X[1]$$

by setting $\hat{n}_T := -h\circ\hat{m}(T)$. This yields a map

$$\hat{N}(T) := \sum_{\sigma\in\Sigma_k}\frac{1}{|\operatorname{Aut}(T)|}\sigma^* n(T)$$

which does not depend on the orientation of T. Consequently we obtain a well-defined map

$$\hat{N}([T]):\mathcal{S}^{(k)}(X[1])\to X[1],\quad \hat{N}([T]) := \sum_{\sigma\in\Sigma_k}\frac{1}{|\operatorname{Aut}(T)|}\sigma^*\hat{n}(T).$$

Finally we set

$$\hat{N}_k : \mathcal{S}^{(k)}(X[1]) \to X[1], \quad \hat{N}_k := id + \sum_{[T]\in[\mathbb{T}](k)} \hat{N}([T])$$

and $N_k := \hat{N}_k \circ i^{\otimes k}$.

PROPOSITION 2.10. *Let (X, d) be a complex equipped with all the structures of Theorem 2.2, i.e. contraction data (i, p, h), a compatible finite filtration $\mathcal{F}_\bullet(X)$ and the structure of a differential graded Lie algebra $(X, D, [\cdot, \cdot])$.*

Then the family of maps $(N_k : \mathcal{S}^{(k)}(H[1]) \to X[1])_{k\in\mathbb{N}}$ introduced in Remark 2.9 defines an L_∞-morphism from H equipped with the induced L_∞-algebra structure to $(X, D, [\cdot, \cdot])$. We call this L_∞-morphism the induced L_∞-morphism*.*

PROOF. Recall the proof of Proposition 2.8 where pointed and truncated decorated oriented trivalent trees were introduced. The claim that $(N_k)_{k\in\mathbb{N}}$ is a morphism of $L_\infty[1]$-algebras from $(H[1], (M_k)_{k\in\mathbb{N}})$ to $(X[1], D, \{\cdot, \cdot\})$ is equivalent to the family of relations

$$\begin{aligned}
& -D(h \circ \hat{M}_k(x_1 \otimes \cdots \otimes x_k)) \\
& +1/2 \sum_{r+s=k} \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma)\{h \circ \hat{M}_r(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}), \cdots \\
& \qquad\qquad \cdots, h \circ \hat{M}_s(x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(k)})\} \\
= \; & ip\hat{M}_k(x_1 \otimes \cdots \otimes x_k) \\
& - \sum_{r+s=k} \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma) h \circ \hat{M}_{(r+1)}(ip \circ \hat{M}_q(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}) \otimes \cdots \\
& \qquad\qquad \cdots \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(k)})
\end{aligned}$$

with $x_1, \ldots, x_k$ arbitrary homogeneous elements of $H[1]$, see Definition 1.12.

Applying $hd + hd = id - ip$ shows that the left-hand side of the above claimed identity is equal to

$$\begin{aligned}
& \Big(-D_r h\hat{M}_k - hd\hat{M}_k - \hat{M}_k + ip\hat{M}_k)\Big) + (\hat{M}_k + D_R h\hat{M}_k) \\
& \quad = \; ip\hat{M}_k - hd\hat{M}_k.
\end{aligned}$$

On the other hand

$$\begin{aligned}
& -(hd\hat{M}_k)(x_1 \otimes \cdots \otimes x_k) \\
= \; & -h\left(\sum_{\sigma\in\Sigma_k} \sum_{[T]\in[\mathbb{T}](k)} \frac{1}{|\operatorname{Aut}(T)|} \sigma^* \hat{m}(P(T))\right)(x_1 \otimes \cdots \otimes x_k) \\
= \; & - \sum_{r+s=k} \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma) h \circ \hat{M}_{(r+1)}(ip \circ \hat{M}_q(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}) \otimes \cdots \\
& \qquad\qquad \cdots \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(k)})
\end{aligned}$$

holds, see the proof of Proposition 2.8. Consequently $(N_k := \hat{N}_k \circ i^{\otimes k})_{k\in\mathbb{N}}$ yields an $L_\infty[1]$-morphism from $(H[1], (M_k)_{k\in\mathbb{N}})$ to $(X[1], D, \{\cdot,\cdot\})$. □

REMARK 2.11. The formulae we gave for the structure maps of the induced L_∞-algebra and of the induced L_∞-morphism can be interpreted as certain Feynman diagrams, see [**Sch1**] for instance.

3. Derived Brackets Formalism

DEFINITION 3.1. A *V-algebra* is a quadruple $(V, [\cdot,\cdot], \mathfrak{a}, \Pi_\mathfrak{a})$ where

(a) $(V, [\cdot,\cdot])$ is a *graded Lie algebra*, i.e. a differential graded Lie algebra with vanishing coboundary operator,
(b) $\mathfrak{a}$ is a graded vector subspace of V and an abelian Lie subalgebra, i.e. $[\mathfrak{a},\mathfrak{a}] = 0$,
(c) $\Pi_\mathfrak{a} : V \to V$ is a projection onto $\mathfrak{a}$ such that the kernel of $\Pi_\mathfrak{a}$ is closed with respect to the graded Lie bracket $[\cdot,\cdot]$.

REMARK 3.2. The term “V-algebra” was introduced in [**CS**] and is an abbreviation for Voronov-algebra. Voronov introduced this objects in [**V1**] where he also initiated the systematic study of higher derived brackets.

DEFINITION 3.3. Let $(V, [\cdot,\cdot], \mathfrak{a}, \Pi_\mathfrak{a})$ be a V-algebra and P a homogeneous element of V of degree $|P|$. The *higher derived brackets* on $\mathfrak{a}$ are given by

$$\begin{array}{rcl} D_k^P : \mathcal{S}^{(k)}(\mathfrak{a}) & \to & a[|P|] \\ x_1 \otimes \cdots \otimes x_k & \mapsto & \Pi_\mathfrak{a}\left([[\ldots[[P,x_1],x_2],\ldots],x_k]\right). \end{array}$$

REMARK 3.4. Because $\mathfrak{a}$ is an abelian Lie subalgebra of $(V, [\cdot,\cdot])$, the higher derived brackets are graded symmetric, i.e.

$$D_k^P(x_1 \otimes \cdots \otimes x_k) = (-1)^{|x_i||x_{(i+1)}|} D_k^P(x_1 \otimes \cdots \otimes x_{(i+1)} \otimes x_i \otimes \cdots \otimes x_k)$$

holds for all homogeneous elements $x_1, \ldots, x_k \in V$.

Suppose P is an element of degree 1. By Definition 1.9 the family of maps $(D_k^P : \mathcal{S}^{(k)}(\mathfrak{a}) \to \mathfrak{a}[1])_{k\in\mathbb{N}}$ comes along with the family of associated Jacobiators $(J_k : \mathcal{S}^{(k)}(\mathfrak{a}) \to \mathfrak{a}[2])_{k\in\mathbb{N}}$.

THEOREM 3.5. *Let $(V, [\cdot,\cdot], \mathfrak{a}, \Pi_\mathfrak{a})$ be a V-algebra and suppose P is a homogeneous element of V of degree 1. Then the family of Jacobiators associated to the higher derived brackets $(D_k^P : \mathcal{S}^{(k)}(\mathfrak{a}) \to \mathfrak{a}[1])_{k\in\mathbb{N}}$ is given by the higher derived brackets $(D_k^{\frac{1}{2}[P,P]} : \mathcal{S}^{(k)}(\mathfrak{a}) \to \mathfrak{a}[2])_{k\in\mathbb{N}}$.*

In particular if P is a Maurer–Cartan element of $(V, [\cdot,\cdot])$, i.e. P is a degree 1 element and satisfies $[P,P] = 0$, then the higher derived brackets $(D_k^P : \mathcal{S}^{(k)}(\mathfrak{a}) \to \mathfrak{a}[1])_{k\in\mathbb{N}}$ equip $\mathfrak{a}$ with the structure of an $L_\infty[1]$-algebra.

PROOF. We refer the interested reader to [**V1**] for the proof. □

REMARK 3.6. One can extend the construction of higher derived brackets from Maurer–Cartan elements to derivations of $(V, [\cdot,\cdot])$ that are not necessarily inner, see [**V2**].

The main application of Theorem 3.5 that we are interested in is the construction of the homotopy Lie algebroid associated to the following data:

- a submanifold S of a Poisson manifold (M, Π) and
- an embedding of the normal bundle of S in M into M as an open neighbourhood of S in M,

see Definition 3.6 in Chapter 3. This construction can be adapted to regular Dirac structures ([**CS**]). Other applications of the higher derived brackets formalism can be found in [**V1**] and [**V2**].

In the application we will consider the Maurer–Cartan element P is not canonically given but depends on certain choices. However one can show that two choices can be related by an automorphism of the graded Lie algebra $(V, [\cdot,\cdot])$. The question arises whether this suffices to related the $L_\infty[1]$-algebra structure on $\mathfrak{a}$ corresponding to two different choices of Maurer–Cartan elements. Theorem 3.7 gives an affirmative answer under certain assumptions that are satisfied in the application we have in mind.

THEOREM 3.7. *Let $(V, [\cdot,\cdot], \mathfrak{a}, \Pi_{\mathfrak{a}})$ be a V-algebra and P a Maurer–Cartan element of $(V, [\cdot,\cdot])$. Moreover all homogeneous components of V are equipped with the structure of topological vector spaces. Suppose $(\phi_t)_{t\in[0,1]}$ is a one-parameter family of automorphisms of $(V, [\cdot,\cdot])$ satisfying*

(a) $\phi_{t=0} = \mathrm{id}$,
(b) $(\phi_t)_{t\in[0,1]}$ *is a solution of the ordinary differential equation*

$$\frac{d}{dt}|_{t=s}\phi_t(\cdot) = [X_s, \phi_s(\cdot)],$$

for all $s \in [0,1]$ where $(X_t)_{t\in[0,1]}$ is a one-parameter family of elements in V_0 such that $\Pi_{\mathfrak{a}}(X_t) = 0$ holds for arbitrary $t \in [0,1]$.

If we assume that the Cauchy problem in $\mathfrak{a}$ given by

$$\frac{d}{dt}|_{t=s}\zeta_s = \Pi_{\mathfrak{a}}([X_s, \zeta_s]), \quad \zeta_0 \in \mathfrak{a}$$

can be solved uniquely and integrates to a one-parameter family $(\Pi_{\mathfrak{a}}\phi_t)_{t\in[0,1]}$ of automorphisms, there is an isomorphism of $L_\infty[1]$-algebras

$$\varphi_t : (\mathfrak{a}, (D_k^P)_{k\in\mathbb{N}}) \xrightarrow{\cong} (\mathfrak{a}, (D_k^{(\phi_t(P))})_{k\in\mathbb{N}})$$

for all $t \in [0,1]$.

REMARK 3.8. Theorem 3.7 was originally proved in [**CS**]. There a generalization to one-parameter family of automorphisms not necessarily generated by *inner* derivations of $(V, [\cdot,\cdot])$ was presented.

We will apply Theorem 3.7 to prove that the homotopy Lie algebroid associated to a submanifold S of a Poisson manifold (M,Π) is unique up to isomorphism, i.e. different choices of embeddings of the normal bundle NS of S in M into M as an open neighbourhood of S lead to isomorphic L_∞-algebra structures on $\Gamma(\wedge NS)$, see Theorem 3.15 in Chapter 3.

Proof. The morphisms $(D_k^{\phi_t(P)}:\mathcal{S}^{(k)}(\mathfrak{a})\to\mathfrak{a}[1])_{k\in\mathbb{N}}$ corresponds to a one-parameter family of codifferentials $Q(t)$ of $(\mathcal{S}^{(k)}(\mathfrak{a}),\Delta)$, see Remark 1.13. Moreover the higher derived brackets $(D_k^{X_t}:\mathcal{S}^{(k)}(\mathfrak{a})\to\mathfrak{a})_{k\in\mathbb{N}}$ yield a one-parameter family of coderivations $M(t)$ of $(\mathcal{S}^{(k)}(\mathfrak{a}),\Delta)$.

We claim that the ordinary differential equation

$$\frac{d}{dt}|_{t=s}Q(t)=M(s)\circ Q(s)-Q(s)\circ M(s)=:[M(s),Q(s)]$$

is satisfied for all $s\in[0,1]$. The formula for $Q(t)$ is given by

$$\begin{aligned}&Q(t)(x_1\otimes\cdots\otimes x_k):=\\&\quad=\sum_{r+s=k}\sum_{\sigma\in(r,s)-\text{shuffles}}\operatorname{sign}(\sigma)D_r^{\phi_t(P)}(x_{\sigma_1}\otimes\cdots x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(k)}.\end{aligned}$$

Because of $\frac{d}{dt}|_{t=s}(\phi_t(P))=[X_s,\phi_s(P)]$ we obtain

$$\begin{aligned}&\frac{d}{dt}|_{t=s}Q(t)(x_1\otimes\cdots\otimes x_k):=\\&\quad=\sum_{r+s=k}\sum_{\sigma\in(r,s)-\text{shuffles}}\operatorname{sign}(\sigma)D_r^{[X_s,\phi_s(P)]}(x_{\sigma_1}\otimes\cdots x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(k)}.\end{aligned}$$

We introduce an auxiliary parameter τ of degree 1 and consider the $\mathbb{R}[\tau]/\tau^2$-module $V[\tau]/\tau^2$. Next we extend the graded Lie bracket $[\cdot,\cdot]$ to a graded Lie bracket on $V[\tau]/\tau^2$ by setting $[\tau x,y]:=\tau[x,y]$ for all $x,y\in V$. The V-algebra structure on $(V,[\cdot,\cdot])$ extends in an $\mathbb{R}[\tau]/\tau^2$-linear manner to a V-algebra structure on $(V[\tau]/\tau^2,[\cdot,\cdot])$. Consider the higher derived brackets $(D_k^{(\phi_t(P)+\tau X_t)})_{k\in\mathbb{N}}$ which come along with a family of associated Jacobiators $(J_k)_{k\in\mathbb{N}}$. By Theorem 3.5 these are given by the higher derived brackets for

$$\frac{1}{2}[\phi_t(P)+\tau X_t,\phi_t(P)+\tau X_t]=\tau[X_t,\phi_t(P)].$$

Therefore the family of morphisms

$$(\frac{\partial}{\partial\tau}|_{\tau=0}J_k(s))_{n\in\mathbb{N}}$$

corresponds to the one-parameter family of coderivations $(\frac{d}{dt}|_{t=s}Q(t))_{s\in[0,1]}$.

We claim that the one-parameter family of coderivations $([M(s),Q(s)])_{s\in[0,1]}$ also corresponds the the one-parameter family morphisms $(\frac{\partial}{\partial\tau}|_{\tau=0}J_k(s))_{n\in\mathbb{N}}$. Recall that the family of Jacobiators is given by

$$
\begin{aligned}
&J_k(s)(x_1\otimes\cdots\otimes x_k) := \\
&\quad = \sum_{r+s=k}\ \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma) D^{(\phi_s(P)+\tau X_s)}_{(s+1)}(D^{(\phi_s(P)+\tau X_s)}_r(x_{\sigma_1}\otimes\cdots\otimes x_r)\otimes\cdots \\
&\qquad\qquad\qquad\qquad\qquad \cdots\otimes x_{(r+1)}\otimes\cdots\otimes x_k).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\frac{d}{d\tau}|_{\tau=0} J_k(s)(x_1\otimes\cdots\otimes x_k) := \\
&\quad = \sum_{r+s=k}\ \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma) D^{X_s}_{(s+1)}(D^{\phi_s(P)}_r(x_{\sigma_1}\otimes\cdots\otimes x_r)\otimes x_{(r+1)}\otimes\cdots\otimes x_k) \\
&\qquad - \sum_{r+s=k}\ \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma) D^{\phi_s(P)}_{(s+1)}(D^{X_s}_r(x_{\sigma_1}\otimes\cdots\otimes x_r)\otimes x_{(r+1)}\otimes\cdots\otimes x_k).
\end{aligned}
$$

It is straightforward to see that this expression corresponds to the coderivation $[M(s),Q(s)]$.

Next we study the solutions of the Cauchy problem given by

$$\frac{d}{dt}|_{t=S}U(t) = M(s)\circ U(s), \quad U(0) = id$$

where $(U(t))_{t\in[0,1]}$ is a one-parameter family of automorphisms of the graded vector space $\mathcal{S}(\mathfrak{a})$. This is equivalent to the family of ordinary differential equations

$$
\begin{aligned}
&\frac{d}{dt}|_{t=s} U_k(t)(x_1\otimes\cdots\otimes x_k) \\
&\quad = \sum_{l=1}^{n}\ \sum_{j_1+\cdots+j_l=k}\ \sum_{\tau\in\Sigma_k} \frac{sign(\tau)}{l!j_1!\cdots j_l!} D^{X_s}_l(U_{j_1}(s)(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma_{j_1}})\otimes\cdots \\
&\qquad\qquad\qquad\qquad \cdots\otimes U_{j_l}(s)(x_{\sigma(j_1+\cdots+j_{(l-1)}+1)}\otimes\cdots\otimes x_{\sigma(k)}))
\end{aligned}
$$

together with the initial conditions $U_1(0) = \mathrm{id}$ and $U_k(0) = 0$ for $k \neq 1$.

First observe that we can consistently set $U_0(t) = 0$ for all $t\in[0,1]$. We claim that there is at most one solution to this family of equations that satisfies $U_0(t) = 0$. The equation for $k = 1$ is simply the Cauchy problem

$$\frac{d}{dt}|_{t=s}U_1(s) = \Pi_{\mathfrak{a}}([X_s, U_1(s)]), \quad U_1(0) = id$$

hence $U_1(t) = \Pi_{\mathfrak{a}}\phi_t$. Now suppose we proved uniqueness of $U_l(t)$ for all $l < k$ and let $U_k(t)$ and $\tilde{U}_k(t)$ be two solutions for the ordinary differential equation above. It follows that the difference $U_k(t) - \tilde{U}_k(t)$ satisfies

$$\frac{d}{dt}|_{t=s}(U_k(t)-\tilde{U}_k(t))(x_1\otimes\cdots\otimes x_k) = \Pi_{\mathfrak{a}}([X_s,(U_k(t)-\tilde{U}_k(t))(x_1\otimes\cdots\otimes x_k)]),$$

and we have $U_k(0)(x_1\otimes\cdots\otimes x_k) - \tilde{U}_k(0)(x_1\otimes\cdots\otimes x_k) = 0$ for all $x_1,\dots,x_k\in\mathfrak{a}$. Consequently $U_k(t)(x_1\otimes\cdots\otimes x_k) = \tilde{U}_k(t)(x_1\otimes\cdots\otimes x_k)$ holds for all $t\in[0,1]$.

It follows that if $(U(t))_{t\in[0,1]}$ exists it is a one-parameter family of automorphisms of the coalgebra because

$$\begin{aligned}&\frac{d}{dt}|_{t=s}(U(t)^{-1}\otimes U(t)^{-1})\circ\Delta\circ U(t)\\&\qquad=-(U(s)^{-1}\otimes U(s)^{-1})((id\otimes M(s)+M(s)\otimes id)\circ\Delta-\Delta\circ M(s))U(s)\end{aligned}$$

vanishes since $(M(t))_{t\in[0,1]}$ is a one-parameter family of coderivations of $(\mathcal{S}(\mathfrak{a}),\Delta)$. Furthermore we define $Z(t):=Q(t)\circ U(t)-U(t)\circ Q(0)$ and calculate

$$\begin{aligned}\frac{d}{dt}|_{t=s}Z(t)&=[M(s),Q(s)]\circ U(s)+Q(s)\circ M(s)\circ U(S)-M(s)\circ U(S)\circ Q(0)\\&=M(s)\circ(Q(s)\circ U(S)-U(s)\circ Q(0))\\&=M(s)\circ Z(s).\end{aligned}$$

The one-parameter family $Z(t)$ corresponds to a family of morphisms $(Z_k(t):\mathcal{S}^{(k)}(\mathfrak{a})\to\mathfrak{a}[1])_{k\in\mathbb{N}}$. A short computation shows that $Z(0)=0$ and $Z_0(t)=\Pi_{\mathfrak{a}}(\phi_tP)-\Pi_{\mathfrak{a}}\phi_t\Pi_{\mathfrak{a}}P$. Because of

$$\frac{d}{dt}|_{t=s}(\Pi_{\mathfrak{a}}\phi_t-\Pi_{\mathfrak{a}}\phi_t\Pi_{\mathfrak{a}})(\cdot)=\Pi_{\mathfrak{a}}([X_s,(\Pi_{\mathfrak{a}}\phi_t-\Pi_{\mathfrak{a}}\phi_t\Pi_{\mathfrak{a}})(\cdot)]$$

and $\Pi_{\mathfrak{a}}\phi_0-\Pi_{\mathfrak{a}}\phi_0\Pi_{\mathfrak{a}}=0$ the component $Z_0(t)$ vanishes for all $t\in[0,1]$. Now the uniqueness of $(Z(t))_{t\in[0,1]}$ follows by the same arguments that were used to prove uniqueness of $(U(t))_{t\in[0,1]}$. Consequently $Z(t)=0$ for all $t\in[0,1]$ which is equivalent to $Q(t)\circ U(t)=U(t)\circ Q(0)$, i.e. $U(t)$ is an isomorphism of complexes from $(\mathcal{S}(\mathfrak{a}),Q(0))$ to $(\mathcal{S}(\mathfrak{a}),Q(t))$ for all $t\in[0,1]$. By Remark 1.13 this is equivalent to the existence of an $L_\infty[1]$-isomorphism

$$\varphi_t:(\mathfrak{a},(D_k^P)_{k\in\mathbb{N}})\xrightarrow{\cong}(\mathfrak{a},(D_k^{(\phi_t(P))})_{k\in\mathbb{N}}).$$

It remains to prove existence of a solution $(U_k(t))_{k\in\mathbb{N}}$ of the family of ordinary differential equations

$$\begin{aligned}&\frac{d}{dt}|_{t=s}U_k(t)(x_1\otimes\cdots\otimes x_k)\\&=\sum_{l=1}^{n}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{sign(\tau)}{l!j_1!\cdots j_l!}D_l^{X_s}(U_{j_1}(s)(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma_{j_1}})\otimes\cdots\\&\qquad\qquad\cdots\otimes U_{j_l}(s)(x_{\sigma(j_1+\cdots+j_{(l-1)}+1))}\otimes\cdots\otimes x_{\sigma(k)}))\end{aligned}$$

with initial conditions $U_1(0)=\mathrm{id}$ and $U^k(0)=0$ for $n\neq1$. We give iterative formulae for $(U_k(t))_{k\in\mathbb{N}}$. For $k=1$ we set $U_1(t):=\Pi_{\mathfrak{a}}\phi_t$ and for $k>1$ we define

$$\begin{aligned}U_k(t)(x_1\otimes\cdots\otimes x_k):=\sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq1}\sum_{j_1+\cdots+j_l=k-1}\frac{1}{kl!j_1!\cdots j_l!}\cdots\\\Pi_{\mathfrak{a}}\big([[\cdots[\phi_t(x_{\sigma(1)}),U_{j_1}(t)(x_{\sigma(2)}\otimes\cdots\otimes x_{\sigma(j_1+1)})],\cdots\\\cdots],U_{j_l}(t)(x_{\sigma(j_1+\cdots+j_{(l-1)}+2)}\otimes\cdots\otimes x_{\sigma(k)})]\big).\end{aligned}$$

The straightforward but cumbersome verification that the family of morphisms $(U_k(t))_{k\in\mathbb{N}}$ actually is a solution of the family of ordinary differential equation is done in Lemma 4 in Chapter 6. That the boundary conditions are satisfied is seens as follows: by definiton $U_1(0) = \pi_{\mathfrak{a}}|_{\mathfrak{a}} = \mathrm{id}$ and if we set $t = 0$ in the iterative formula for $U_k(t)$ we see that because of $\phi_0 = \mathrm{id}$ all the terms on the right-hand side are in $\mathfrak{a}$. But $\mathfrak{a}$ is an abelian Lie subalgebra, so all the terms on the right-hand side vanish and hence so does $U_k(0)$ for $k > 1$. □

4. Smooth graded Manifolds

Definition 4.1. A *unital graded commutative associative algebra* is a triple $(V, 1, \cdot)$ where V is a graded vector space, $1 \in V_0$ and $\cdot$ is a morphism of graded vector spaces

$$V \otimes V \to V$$

such that

(a) $a \cdot b = (-1)^{|a||b|} b \cdot a$,
(b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and
(c) $a \cdot 1 = a = 1 \cdot a$

hold for all homogeneous $a, b, c \in V$.

A morphism $\varphi : V \to V[k]$ of graded vector spaces is a *graded derivation of degree* k of the unital graded commutative algebra $(V, 1, \cdot)$ if

$$\varphi(x \cdot y) = \varphi(x) \cdot y + (-1)^{k|x|} x \cdot \varphi(x)$$

holds for all homogeneous $x, y \in V$. A *derivation* of $(V, 1, \cdot)$ is an element of the graded vector space $\underline{\mathrm{Hom}}(V, V)$ – see Definition 1.1 in Section 1 – whose k'th component is a graded derivation of degree k. Derivations form a graded subvector space of the graded vector space $\underline{\mathrm{Hom}}(V, V)$ which we denote by $\mathrm{Der}(V)$.

A *morphism* ϕ from the unital graded commutative associative algebra $(V, 1_V, \cdot_V)$ to the unital graded commutative associative algebra $(W, 1_W, \cdot_W)$ is a morphism ϕ of graded vector spaces from V to W which maps 1_V to 1_W such that

$$\phi(x \cdot_V y) = \phi(x) \cdot_W \phi(y)$$

holds for all x and y in V.

Given two unital graded commutative associative algebras $(V, 1_V, \cdot_V)$ and $(W, 1_W, \cdot_W)$ the tensor product $V \otimes W$ inherits the structure of a unital graded commutative associative algebra by setting $1_{V\otimes W} := 1_V \otimes 1_W$ and

$$(v_1 \otimes w_1) \cdot_{V\otimes W} (v_2 \otimes w_w) := (-1)^{|w_1||v_2|} (v_1 \cdot_V v_2) \otimes (w_1 \cdot_W w_2)$$

for all homogeneous elements v_1, v_2 in V and w_1, w_2 in W.

Remark 4.2. Given a graded vector space V, the graded symmetric algebra $\mathcal{S}(V)$ is a unital graded commutative associative algebra with product inherited from the associative product on the tensor algebra $\mathcal{T}(V)$ given by

$$(x_1 \otimes \cdots \otimes x_k) \otimes (y_1 \otimes \cdots \otimes y_l) \mapsto x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l.$$

Definition 4.3. Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra. A *module* over $(V, 1, \cdot)$ is a pair (M, ϕ) where M is a graded vector space and ϕ is a morphism of graded vector spaces $\phi : V \to \underline{\mathrm{Hom}}(M, M)$ such that $\phi(1) = \mathrm{id}$ is satisfied and

$$\phi(v \cdot w)(m) = (\phi(v) \circ \phi(w))(m)$$

holds for all $v, w \in V$ and $m \in M$.

Let (M, ϕ) be a module over the unital graded commutative associative algebra $(V, 1, [\cdot, \cdot])$. A morphism $\alpha : V \to M[k]$ of graded vector spaces is a *graded derivation of V with values in M of degree k* if

$$\alpha(x \cdot y) = (-1)^{|x||k|}\phi(x)(\alpha(y)) + (-1)^{|y|(|x|+k)}\phi(y)(\alpha(x))$$

holds for all homogeneous $x, y \in V$. A *derivation of* $(V, 1, \cdot)$ *with values in* M is an element of the graded vector space $\underline{\mathrm{Hom}}(V, M)$ whose k'th component is a graded derivation of V with values in M of degree k. Derivations form a graded subvector space of the graded vector space $\underline{\mathrm{Hom}}(V, M)$ which we denote by $\mathrm{Der}(V, M)$.

Remark 4.4. Observe that any unital graded commutative associative algebra $(V, 1, \cdot)$ is a module over itself via

$$\mathrm{ad} : x \mapsto x \cdot .$$

The definition of derivations of $(V, 1, \cdot)$ coincides with the definition of derivations of $(V, 1, \cdot)$ with values in the module (V, ad).

Suppose (M, ϕ) is a module over $(V, 1, \cdot)$. Observe that $\mathrm{Der}(V, M)$ inherits the structure of a module over $(V, 1, \cdot)$ by

$$\hat{\phi} : V \to \mathrm{Der}(V, M), \quad x \mapsto (f \mapsto x \cdot f).$$

From now on we will always consider $\mathrm{Der}(V, M)$ as a module over $(V, 1, \cdot)$.

Definition 4.5. Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra. The space of *multiderivations* $\mathcal{D}(V)$ of $(V, 1, \cdot)$ is the graded vector space

$$\mathcal{D}(V) := \mathcal{S}_V(\mathrm{Der}(V)[-1]),$$

i.e. the graded symmetric algebra generated by $\mathrm{Der}(V, M)[-1]$ as a module over V.

The *derivation degree* on $\mathcal{D}(V)$ is given by $\mathcal{D}^{(k)}(V) := \mathcal{S}_V^{(k)}(\mathrm{Der}(V, M)[-1])$.

Remark 4.6. The graded vector space $\mathcal{D}(V)$ of multiderivations of a unital graded commutative associative algebra $(V, 1, \cdot)$ inherits a module structure from $\mathrm{Der}(V)$. Moreover it is a unital graded commutative associative algebra, see Remark 4.2.

DEFINITION 4.7. Let M be a smooth finite dimensional manifold.

A *graded vector bundle* $E_\bullet$ over M is a collection $(E_i)_{i\in\mathbb{Z}}$ of finite rank vector bundles over M. A graded vector bundle $E_\bullet$ over M is *bounded* if $E_k = \{0\}$ for all k smaller than some l and larger than some L. Since we only consider bounded graded vector bundles we will drop the adjective bounded from now on.

The *dual* $E_\bullet^*$ of a graded vector bundle $E_\bullet$ over M is the graded vector bundle given by $(E_{-i}^*)_{i\in\mathbb{Z}}$. The *n'th suspension* $E[n]_\bullet$ of a graded vector bundle $E_\bullet$ is the graded vector bundle given by $(E_{i+n})_{i\in\mathbb{Z}}$.

The *algebra of smooth functions* on a graded vector bundle $E_\bullet$ over M is the unital graded commutative associative algebra

$$\mathcal{C}^\infty(E_\bullet) := \Gamma(\mathcal{S}(E_\bullet^*)).$$

A *morphism* from the graded vector bundle $E_\bullet$ to the graded vector bundle $F_\bullet$ is a morphism in the category of unital graded commutative associative algebras from $\mathcal{C}^\infty(F_\bullet)$ to $\mathcal{C}^\infty(E_\bullet)$.

DEFINITION 4.8. A *smooth graded manifold* $\mathcal{M}$ with body M is a unital graded commutative associative algebra $\mathcal{A}_\mathcal{M}$ that is isomorphic to $\mathcal{C}^\infty(E_\bullet)$ for some graded vector bundle $E_\bullet$ over M. The *algebra of smooth functions* $\mathcal{C}^\infty(\mathcal{M})$ on $\mathcal{M}$ is the unital graded commutative associative algebra $\mathcal{A}_\mathcal{M}$. The *algebra of multivector fields* $\mathcal{V}(\mathcal{M})$ on $\mathcal{M}$ is the $\mathcal{C}^\infty(\mathcal{M})$ module $\mathcal{D}(\mathcal{C}^\infty(\mathcal{M}))$. A *k-vector field* on $\mathcal{M}$ is an element of $\mathcal{D}^{(k)}(\mathcal{C}^\infty(\mathcal{M}))$.

A *morphism* from the smooth graded manifold $\mathcal{M}$ to the smooth graded manifold $\mathcal{N}$ is a morphism of unital graded commutative algebras from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\mathcal{M})$.

REMARK 4.9. Observe that the specific isomorphism between $\mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{C}^\infty(E_\bullet)$ is *not* part of the data that define a smooth graded manifold. Moreover we remark that we do not follow the more general definition of graded manifolds in terms of sheaves of locally free graded algebras since for all applications we will consider the definition we give suffices and is in fact more convenient to work with.

LEMMA 4.10. *Let $E_\bullet$ be a graded vector bundle over M. The unital graded commutative symmetric algebra $\mathcal{V}(E_\bullet)$ is the algebra of smooth functions on the total space of the graded vector bundle $T^*[1]E_\bullet$ over M.*

Consequently the algebra of multivector fields $\mathcal{V}(\mathcal{M})$ on an arbitrary smooth graded manifold $\mathcal{M}$ is the algebra of smooth functions of a smooth graded manifold which we denote by $T^*[1]\mathcal{M}$.

REMARK 4.11. The observation that $\mathcal{V}(\mathcal{M})$ equals $\mathcal{C}^\infty(T^*[1]\mathcal{M})$ has an important consequence: since $T^*[1]\mathcal{M}$ is a cotangent bundle (with a degree shift) it should carry the graded analogon of a symplectic form. In particular the algebra of smooth functions on $T^*[1]\mathcal{M}$ should be some sort of graded version of a Poisson algebra. This is indeed the case. However, instead of explaining this in terms of the theory of graded symplectic manifolds, we will use a more algebraic approach.

Proof. Consider the space of derivations $\mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*)))$. Restriction of any such derivation to $\Gamma(\mathcal{S}^{(0)}(E_\bullet^*)) = \mathcal{C}^\infty(M)$ yields a derivation of $\mathcal{C}^\infty(M)$ with values in $\Gamma(\mathcal{S}(E_\bullet^*))$, i.e. we obtain a morphism of graded vector spaces

$$\mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*))) \to \mathrm{Der}(\mathcal{C}^\infty(M), \Gamma(\mathcal{S}(E_\bullet^*))).$$

Any choice of a family of connections $\nabla_\bullet$ on $E_\bullet$ yields a left inverse to this morphism, hence it is surjective. The kernel of

$$\mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*))) \to \mathrm{Der}(\mathcal{C}^\infty(M), \Gamma(\mathcal{S}(E_\bullet^*)))$$

consists of all those derivations of $\Gamma(\mathcal{S}(E_\bullet^*))$ that are $\mathcal{C}^\infty(M)$-linear. Hence we have a short exact sequence of graded vector spaces

$$0 \to \mathrm{Der}_{\mathcal{C}^\infty(M)}(\Gamma(\mathcal{S}(E_\bullet^*))) \to \mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*))) \to \mathrm{Der}(\mathcal{C}^\infty(M), \Gamma(\mathcal{S}(E_\bullet^*))) \to 0.$$

The fact that $\mathrm{Der}(\mathcal{C}^\infty(M)) \cong \Gamma(TM)$ (see [**Mi**] for instance) generalizes to

$$\mathrm{Der}(\mathcal{C}^\infty(M), \Gamma(\mathcal{S}(E_\bullet^*))) \cong \Gamma(TM \otimes \mathcal{S}(E_\bullet^*)).$$

Moreover $\mathrm{Der}_{\mathcal{C}^\infty(M)}(\Gamma(\mathcal{S}(E_\bullet^*)))$ is isomorphic to $\Gamma(\mathrm{Der}(\mathcal{S}(E_\bullet^*))) \cong \Gamma(E \otimes \mathcal{S}(E_\bullet^*))$. Consequently we obtain a short exact sequence

$$0 \to \Gamma(E \otimes \mathcal{S}(E_\bullet^*)) \to \mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*))) \to \Gamma(TM \otimes \mathcal{S}(E_\bullet^*)) \to 0.$$

We choose a family of connections $\nabla_\bullet$ on the graded vector bundle $E_\bullet$. This provides a splitting of the above exact sequence, i.e.

$$\mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*))) \cong \Gamma((E \oplus TM) \otimes \mathcal{S}(E_\bullet^*))$$

as graded vector spaces and as modules over $\Gamma(\mathcal{S}(E_\bullet^*))$ as can be checked easily.

This yields

$$\mathcal{V}(E_\bullet) = \mathcal{S}_{\Gamma(\mathcal{S}(E_\bullet^*))}(\mathrm{Der}(\Gamma(\mathcal{S}(E_\bullet^*)))[-1]) \cong \mathcal{S}_{\Gamma(\mathcal{S}(E_\bullet^*))}(\Gamma((E \oplus TM)[-1] \otimes \mathcal{S}(E_\bullet^*))).$$

Since we take the symmetric product over $\Gamma(\mathcal{S}(E_\bullet^*))$, it is in particular $\mathcal{C}^\infty(M)$-linear and consequently the symmetric algebra generated by the $\Gamma(\mathcal{S}(E_\bullet^*))$ module $\Gamma((E \oplus TM)[-1] \oplus \mathcal{S}(E_\bullet^*))$ is the space of sections of the bundle $\mathcal{S}_{\mathcal{S}(E_\bullet^*)}((E \oplus TM)[-1] \otimes \mathcal{S}(E_\bullet^*))$. Using the family of connections $\nabla_\bullet$ on $E_\bullet$ we obtain an isomorphism of bundles $\mathcal{S}(T^*[1]E)^* \cong \mathcal{S}_{\mathcal{S}(E_\bullet^*)}((E \oplus TM)[-1] \otimes \mathcal{S}(E_\bullet^*))$ and the induced isomorphism on sections is exactly inverse to

$$\Gamma(\mathcal{S}_{\mathcal{S}(E_\bullet^*)}((E \oplus TM)[-1] \otimes \mathcal{S}(E_\bullet^*)) \cong \mathcal{V}(E_\bullet).$$

Hence we conclude with $\mathcal{C}^\infty(T^*[1]E) = \mathcal{V}(E_\bullet)$. $\square$

Definition 4.12. Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra. A *graded Poisson bracket of degree* k on $(V, 1, \cdot)$ is a morphism $[\cdot,\cdot]$ of graded vector spaces $V \otimes V \to V[-k]$ such that

(a) $(V[k], [\cdot,\cdot])$ is a graded Lie algebra, see Definition 3.1 and
(b) for any $X \in V_l$ the morphism $[X, \cdot] : V \to V[l-k]$ is a graded derivation of $(V, 1, \cdot)$ of degree $(l-k)$.

REMARK 4.13. If we talk about a *graded Poisson bracket* without further specification we refer to a graded Poisson bracket of degree 0.

A graded Poisson bracket of degree 1 is also known as a *Gerstenhaber bracket*.

LEMMA 4.14. *Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra. The unital graded commutative associative algebra of multiderivations $\mathcal{D}(V)$ carries a Gerstenhaber bracket $[\cdot,\cdot]_{SN}$ such that*

(a) *for $f,\ g \in \mathcal{D}^{(0)}(V) = V$ the bracket is given by*
$$[f, g]_{SN} = 0,$$
(b) *for homogeneous $f \in V$ and $X \in \mathcal{D}^{(1)}(V)[1] = \mathrm{Der}(V)$ the bracket is given by*
$$[X, f]_{SN} = X(f) = -(-1)^{|X|(|f|-1)}[f, X]_{SN} \quad \textit{and}$$
(c) *for homogeneous $X,\ Y$ in $\mathrm{Der}(V)$ the bracket is given by*
$$[X, Y]_{SN} = X \circ Y - (-1)^{|X||Y|} Y \circ X.$$

The bracket $[\cdot,\cdot]_{SN}$ is known as the Schouten-Nijenhuis bracket.

PROOF. Since $\mathcal{D}(V)$ is generated as an algebra by V and $\mathrm{Der}(V)[-1]$, it is sufficient to describe its restriction to $V \oplus \mathrm{Der}(V)[-1]$. It is straightforward to check that the graded Jacobi identity (see Definition 1.11) is satisfied, i.e.
$$[X, [Y, f]_{SN}]_{SN} = [[X, Y]_{SN}, f]_{SN}] + (-1)^{|X||Y|}[Y, [X, f]_{SN}]_{SN},$$
$$[X, [Y, Z]_{SN}]_{SN} = [[X, Y]_{SN}, Z]_{SN}] + (-1)^{|X||Y|}[Y, [X, Z]_{SN}]_{SN}$$
holds for all homogeneous $X, Y, Z \in \mathcal{D}^{(1)}(V)[1] = \mathrm{Der}(V)$ and $f \in V$.

Let X be a homogeneous element of $\mathrm{Der}(V)$ and f a homogeneous element of V. Extend $[X, \cdot]_{SN}$ and $[f, \cdot]_{SN}$ to graded biderivations of $\mathcal{D}(V)$ of degree $|X|$ and $|f| - 1$ respectively. This yields an operation from $(V[1] \oplus \mathrm{Der}(V)) \otimes \mathcal{D}(V)[1]$ to $\mathcal{D}(V)[1]$. Finally extend this by graded skew-symmetry to an operation $[\cdot,\cdot]_{SN}$ from $\mathcal{D}(V)[1] \otimes \mathcal{D}(V)[1]$ to $\mathcal{D}(V)[1]$. By construction $[\cdot,\cdot]_{SN}$ will also satisfy the graded Jacobi-identity. □

DEFINITION 4.15. Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra. An L_∞-algebra $(V, (\lambda_k)_{k\in\mathbb{N}})$ is a P_∞*-algebra* if for arbitrary homogeneous $x_1, \ldots, x_{(k-1)}$ in V the morphism
$$\lambda_k(x_1 \otimes \cdots \otimes x_{(k-1)} \otimes \cdot) : V \to V[2 - k + |x_1| + \cdots + |x_{(k-1)}|]$$
of graded vector spaces is a graded derivation of $(V, 1, \cdot)$ of degree $2 - k + |x_1| + \cdots + |x_{(k-1)}|$.

REMARK 4.16. The term P_∞-algebra was introduced in [**CF**]. Observe that not all algebras over a cofibrant resolution of the Poisson-operad yield P_∞-algebras. However, the definition given above suffices for our purposes.

Since L_∞-algebras are by definition graded skew-symmetric, the defining relation of a P_∞-algebra implies that the structure maps are graded multiderivations. In particular if $(V, (\lambda_k)_{k\in\mathbb{N}})$ is a P_∞-algebra

$$\begin{aligned}
&\lambda_k(x_1 \otimes \cdots \otimes x_{(i-1)} \otimes (y \cdot z) \otimes x_i \otimes x_k) = \\
&\quad = \quad (-1)^{(|x_1|+\cdots+|x_{(i-1)}|+k-2)|y|} y \cdot \lambda_k(x_1 \otimes \cdots \otimes x_{(i-1)} \otimes z \otimes x_i \otimes x_k) \\
&\qquad\quad +(-1)^{(|x_1|+\cdots+|x_{(i-1)}|+|y|+k-2)|z|} z \cdot \lambda_k(x_1 \otimes \cdots \otimes x_{(i-1)} \otimes y \otimes x_i \otimes x_k)
\end{aligned}$$

holds for all homogeneous $x_1, \ldots, x_{(k-1)}, y, z \in V$.

In Section 1 we saw that differential graded Lie algebras are special cases of L_∞-algebras. Similarly differential graded Poisson algebras are those P_∞-algebras where all structure maps $(\lambda_k)_{k\in\mathbb{N}}$ except for $k = 1$ and 2 vanish.

Definition 4.17. Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra. A *differential graded Poisson algebra* is a triple $(V, d, [\cdot,\cdot])$ where

(a) $(V, d, [\cdot,\cdot])$ is a differential graded Lie algebra,
(b) d is a derivation of $(V, 1, \cdot)$ degree 1 and
(c) $(V, [\cdot,\cdot])$ is a graded Poisson algebra (of degree 0).

Lemma 4.18. *Let $(V, 1, \cdot)$ be a unital graded commutative associative algebra.*

The quadruple $(\mathcal{D}(V)[1], [\cdot,\cdot]_{SN}, V[1], \mathrm{pr}_V[1])$ is a V-algebra, see Definition 3.1. Here pr_V denotes the projection $\mathcal{D}(V) \to \mathcal{D}^{(0)}(V) = V$.

If $P \in \mathcal{D}(V)$ is an element of total degree 2 that satisfies $[P, P]_{SN} = 0$, the higher derived brackets

$$D^P_k(v_1 \otimes \cdots \otimes v_k) = \mathrm{pr}_V([[\cdots[[P, v_1]_{SN}, v_2]_{SN}, \cdots], v_k]), k \in \mathbb{N},$$

equip V with the structure of a P_∞-algebra.

Proof. By definition V is an abelian Lie subalgebra of $(\mathcal{D}(V)[1], [\cdot,\cdot]_{SN})$. Moreover the kernel of pr_V is $\mathcal{D}^{(\geq 1)}(V)$ and since $[\cdot,\cdot]_{SN}$ maps $\mathcal{D}^{(\geq k)}(V) \otimes \mathcal{D}^{(\geq l)}(V)$ to $\mathcal{D}^{(\geq k+l-1)}(V)$, $\mathcal{D}^{(\geq 1)}(V)$ is closed under $[\cdot,\cdot]_{SN}$.

Theorem 3.5 implies that $(V[1], (D^P_k)_{k\in\mathbb{N}})$ is an $L_\infty[1]$-algebra – equivalently V is equipped with an L_∞-algebra structure. That the structure maps $(D^P_k)_{k\in\mathbb{N}}$ satisfy the derivation property follows from the identity

$$[X, Y \cdot Z]_{SN} = [X, Y]_{SN} \cdot Z + (-1)^{|Y||Z|}[X, Z]_{SN} \cdot Y$$

for homogeneous X, Y, Z in $\mathcal{D}(V)$ and the fact that $\mathrm{pr}_V : \mathcal{D}(V) \to V$ is a morphism of unital graded commutative associative algebras. □

Definition 4.19. Let $\mathcal{M}$ be a smooth graded manifold.

A *Poisson multivector field* on $\mathcal{M}$ is Maurer–Cartan element of $(\mathcal{V}(\mathcal{M})[1], [\cdot,\cdot]_{SN})$, i.e. an element P of total degree 2 in $\mathcal{V}(\mathcal{M})$ that satisfies $[P, P]_{SN} = 0$.

A *Poisson k-vector field* on $\mathcal{M}$ is a Poisson multivector field on $\mathcal{M}$ that lies in $\mathcal{V}^{(k)}(\mathcal{M})$. In particular a *Poisson bivector field* on $\mathcal{M}$ is an element P of $\mathcal{V}^{(2)}(\mathcal{M})$ that satisfies $[P, P]_{SN} = 0$.

Corollary 4.20. *Let $\mathcal{M}$ be a smooth graded manifold equipped with a Poisson multivector field P. Then the higher derived brackets $(D_k^P)_{k\in\mathbb{N}}$ equip the algebra of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ on $\mathcal{M}$ with the structure of a P_∞-algebra.*

Proof. This is an immediate consequence of Lemma 4.18. $\square$

CHAPTER 3

Coisotropic Submanifolds

In Section 1 we review basic definitions and facts from Poisson geometry. Section 2 contains a rather detailed introduction to coisotropic submanifolds. In particular we define the Lie algebroid complex associated to a coisotropic submanifold, introduce the quotient space and the quotient algebra and establish the connection between the zero'th Lie algebroid cohomology and the quotient algebra. Finally we explain how to enrich the Lie algebroid complex by higher structure maps. This procedure makes use of the higher derived brackets formalism – see Section 3, Chapter 2 – and yields an L_∞-algebra structure known as the homotopy Lie algebroid. We essentially follow [**OP**] and [**CF**]. This L_∞-algebra depends on a choice of an embedding of the normal bundle of the submanifold under consideration into the ambient manifold. Theorem 3.7 asserts that different choices lead to isomorphic L_∞-algebras. It was first presented in [**OP**] for coisotropic submanifolds of symplectic manifolds and extended to arbitrary submanifolds of Poisson manifolds in the joint paper [**CS**] with Cattaneo.

1. Poisson Manifolds

Let M be a smooth finite-dimensional manifold.

Definition 1.1. A bivector field $\Pi \in \mathcal{V}^2(M) = \Gamma(\wedge^2 TM)$ on M is a *Poisson bivector field* if it is a Maurer–Cartan element of the graded Lie algebra $(\mathcal{V}(M)[1], [\cdot,\cdot]_{SN})$ – see Lemma 4.18, Chapter 2 – i.e. if

$$[\Pi,\Pi]_{SN} = 0$$

is satisfied.

A pair (M,Π) with M a smooth finite-dimensional manifold and Π a Poisson bivector field on M is called a *Poisson manifold*.

Example 1.2. (a) $0 \in \Gamma(\wedge TM)$ is a Poisson structure for arbitrary M.

(b) Let Σ be a two dimensional manifold. Since $\Gamma(\wedge^3 TM) = \{0\}$, any bivector field on Σ is a Poisson bivector field.

(c) Consider a smooth function f from an open subset U of $\mathbb{R}^3$ to $\mathbb{R}$. Then

$$\Pi_f := \frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right) + \frac{\partial f}{\partial z}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}\right)$$

is a Poisson bivector field on U.

(d) Let $\mathfrak{g}$ be a finite dimensional vector space over $\mathbb{R}$ and

$$[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$$

a skew symmetric bilinear map. This map may also be interpreted as an element of $\mathfrak{g} \otimes \wedge^2(\mathfrak{g}^*)$, i.e. as an linear bivector field Z on $\mathfrak{g}^*$. It is a straightforward calculation in a basis that $[Z, Z]_{SN}$ is equal to the linear trivector field associated to the Jacobiator

$$J(x, y, z) := [x, [y, z]] + [z, [x, y]] + [y, [z, x]].$$

Consequently, Z is a Poisson bivector field if and only if $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra.

(e) Let M be a manifold and ω a two-form on M such that the vector bundle map

$$\omega^{\#} : TM \to T^*M$$

given by contraction with ω is an isomorphism. Then

$$(\omega^{\#})^{-1} : T^*M \to TM$$

is skew self-adjoint and hence can be interpreted as a bivector field which we denote by ω^{-1}. A short calculation in local coordinates shows that $[\omega^{-1}, \omega^{-1}]_{SN}$ is equal to minus the vector field given by $(\wedge^3\omega^{\#})(d_{DR}\omega)$. Consequently ω^{-1} is a Poisson bivector field if and only ω is closed with respect to the de Rham differential, i.e. (M, ω) is a symplectic manifold.

Remark 1.3. As observed in Lemma 4.18 in Chapter 2, the triple

$$((\mathcal{V}(M)[1], [\cdot,\cdot]_{SN}), \mathcal{C}^\infty(M)[1], \mathrm{pr}[1])$$

is a V-algebra, see Definition 3.1 in Chapter 2. By Definition a Poisson bivector field Π is a Maurer–Cartan element of $(\mathcal{V}(M)[1], [\cdot,\cdot]_{SN})$, hence it yields a P_∞-algebra structure on $\mathcal{C}^\infty(M)$. The only nontrivial higher derived bracket is the binary operation given by

$$(f, g) \mapsto -[[\Pi, f]_{SN}, g]_{SN}.$$

The minus sign is an effect of the décalage-isomorphism.

Lemma 1.4. *Given a Poisson manifold* (M, Π)*, the operation*

$$\{f, g\}_\Pi := -[[\Pi, f]_{SN}, g]_{SN}$$

equips the algebra $\mathcal{C}^\infty(M)$ *with the structure of a Poisson algebra (i.e. a graded Poisson algebra of degree* 0 *concentrated in degree* 0*).*

We refer to $\{\cdot,\cdot\}_\Pi$ *as the* Poisson bracket *associated to the Poisson bivector field* Π.

PROOF. As observed in Remark 1.3 the operation $\{\cdot,\cdot\}_\Pi$ can be seen as the higher derived bracket associated to the Maurer–Cartan element Π of the V-algebra $(\mathcal{V}(M)[1],[\cdot,\cdot]_{SN},\mathcal{C}^\infty(M)[1],\mathrm{pr}[1])$. The vanishing of the Jacobiators associated to $(\{\cdot,\cdot\}_\Pi)$ reduces to the Jacobi identity and the derivation property introduced in Definition 4.15 in Chapter 2 translates into the fact that $\{\cdot,\cdot\}_\Pi$ is a biderivation with respect to the multiplication on $\mathcal{C}^\infty(M)$. $\square$

REMARK 1.5. In fact, the process $\Pi \mapsto \{\cdot,\cdot\}_\Pi$ can be reversed: *every* biderivation $\{\cdot,\cdot\}$ of $\mathcal{C}^\infty(M)$ is given as the derived bracket associated to some bivector field Z. The calculation

$$\begin{aligned}
\{f,\{g,h\}_Z\}_Z &= [[Z,f]_{SN},[[Z,g]_{SN},h]_{SN}]_{SN} = \\
&= [[[Z,f]_{SN},[Z,g]_{SN}],h]_{SN} + [[Z,g]_{SN},[[Z,f]_{SN},h]_{SN}]_{SN} \\
&= [[Z,[f,[Z,g]_{SN}]_{SN}]_{SN},h]_{SN} + [[f,[Z,[Z,g]_{SN}]_{SN}]_{SN},h]_{SN} \\
&\quad\ +[[Z,g]_{SN},[[Z,f]_{SN},h]_{SN}]_{SN} \\
&= -[[Z,[[Z,g]_{SN},f]_{SN}]_{SN},h]_{SN} + \frac{1}{2}[[f,[[Z,Z]_{SN},g]_{SN}]_{SN},h]_{SN} \\
&\quad\ +[[Z,g]_{SN},[[Z,f]_{SN},h]_{SN}]_{SN} \\
&= -\{\{g,f\}_Z,h\}_Z + \{g,\{f,h\}_Z\}_Z - \frac{1}{2}[[[[Z,Z]_{SN},g]_{SN},f]_{SN},h]_{SN}
\end{aligned}$$

yields

$$\{f,\{g,h\}_Z\}_Z - \{\{f,g\}_Z,h\}_Z - \{g,\{f,h\}_Z\}_Z = \frac{1}{2}[[[[Z,Z]_{SN},f]_{SN},g]_{SN},h]_{SN}.$$

Non-degeneracy of $[\cdot,\cdot]_{SN}$ and the Jacobi identity for $\{\cdot,\cdot\}_Z$ force $[Z,Z]_{SN}$ to vanish.

DEFINITION 1.6. Let (M,Π) be a Poisson manifold. The *Hamiltonian vector field* X_f associated to $f \in \mathcal{C}^\infty(M)$ is given by

$$X_f := -[\Pi,f]_{SN}.$$

This yields a vector bundle map $\Pi^\# : T^*M \to TM$: locally any element $\varphi \in T^*_xM$ can be written as $d_{DR}f|_x$ for some locally defined smooth function f on M. We set

$$\Pi^\#(\varphi) := X_f|_x.$$

LEMMA 1.7. *The description of $\Pi^\#$ given above yields a well-defined vector bundle map.*

PROOF. Let f and g be two locally defined smooth functions satisfying

$$d_{DR}f|_x = \varphi = d_{DR}g|_x.$$

By changing f to $f-f(0)$ and g to $g-g(0)$ we can assume without loss of generality that $f(x)=0=g(x)$. Then $d_{DR}f|_x = d_{DR}g|_x$ implies that $(f-g)$

vanishes at least quadratically in x (quadratically with respect to some chart – and equivalently: any chart). In local coordinates we have

$$X_f - X_g = -[\Pi, (f-g)]_{SN} = \sum_{i,j} \Pi^{ij} \frac{\partial(f-g)}{\partial x^i} \frac{\partial}{\partial x^j}$$

and consequently $X_f|_x = X_g|_x$.
Linearity of $\Pi^{\#}$ is straightforward. □

REMARK 1.8. The Poisson bracket $\{\cdot,\cdot\}_\Pi$ associated to a Poisson bivector field Π on M can be written as

$$\{f,g\}_\Pi = [\Pi^{\#}(d_{DR}f), g]_{SN} = < \Pi^{\#}(d_{DR}f), d_{DR}g > .$$

Here $< \cdot,\cdot >$ denotes the pairing between vector fields and one-forms on M induced by the contraction between TM and T^*M.

DEFINITION 1.9. Let (M,Π) be a Poisson manifold. The *group of automorphisms* $\mathrm{Aut}(M,\Pi)$ of the Poisson algebra $(\mathcal{C}^\infty(M), \{\cdot,\cdot\}_\Pi)$ is given by all invertible morphisms $\Psi : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ of $\mathbb{R}$-modules which satisfy

(a) $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g)$ and
(b) $\Psi(\{f,g\}_\Pi) = \{\Psi(f), \Psi(g)\}_\Pi$

for arbitrary $f, g \in \mathcal{C}^\infty(M)$.

LEMMA 1.10. *The map that associates to* $\varphi \in \mathrm{Diff}(M)$ *the algebra automorphism*

$$\varphi^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad f \mapsto f \circ \varphi$$

induces an isomorphism between the group of Poisson diffeomorphisms $\mathrm{Diff}_\Pi(M)$*, i.e. diffeomorphisms* φ *satisfying*

$$\varphi^*(\{f,g\}_\Pi) = \{\varphi^*(f), \varphi^*(g)\}_\Pi$$

and $\mathrm{Aut}(M,\Pi)$*.*

PROOF. The proof relies on the fact that for every automorphisms Ψ of the unital algebra $\mathcal{C}^\infty(M)$ there is a unique $\varphi \in \mathrm{Diff}(M)$ such that $\Psi = \varphi^*$, see [**AMR**] for instance. □

DEFINITION 1.11. Let (M,Π) be a Poisson manifold. A vector field $X \in \Gamma(TM)$ is *Poisson* if $[X,\Pi]_{SN} = \mathcal{L}_X(\Pi) = 0$.

LEMMA 1.12. *Every Hamiltonian vector field is a Poisson vector field.*

PROOF. A Hamiltonian vector field is by definition equal to $-[\Pi, f]_{SN}$ for some smooth function f. The calculation

$$[-[\Pi,f]_{SN}, \Pi]_{SN} = [\Pi, [\Pi,f]_{SN}]_{SN} = [[\Pi,\Pi]_{SN}, f]_{SN} - [\Pi,[\Pi,f]_{SN}]_{SN}$$

implies that $[-[\Pi,f]_{SN}, \Pi]_{SN}$ vanishes. □

LEMMA 1.13. *Suppose (M,Π) is a Poisson manifold. Let $(X_t)_{t\in[0,1]}$ be a smooth one-parameter family of Poisson vector fields, i.e. a section of the pull back of $TM \to M$ along $M \times [0,1] \to M$, such that its restriction to $M \times \{s\} \cong M$ is a Poisson bivector field. Assume that $(\varphi_t)_{t\in[0,1]}$ is the smooth one-parameter family of diffeomorphisms generated by $(X_t)_{t\in[0,1]}$, i.e.*

(a) *there is a smooth map $\hat{\varphi} : M \times [0,1] \to M$ such that the composition with $M \cong M \times \{t\} \hookrightarrow M \times [0,1]$ is equal to φ_t for arbitrary $t \in [0,1]$,*
(b) *$\varphi_0 = \mathrm{id}_M$ and*
(b) *$\frac{d}{dt}|_{t=s}\varphi_t = X_s|_{\varphi_s}$ holds for arbitrary $s \in [0,1]$.*

Then φ_s is a Poisson-diffeomorphism for all $s \in [0,1]$.

PROOF. The flow equation above is equivalent to the following equation

$$\frac{d}{dt}|_{t=s}\varphi_t^*(\cdot) = \varphi_s^*([X_s,\cdot]_{SN})$$

for arbitrary $s \in [0,1]$, see [**Mi**] for instance. Here $\varphi_s^*(\cdot) := T(\varphi_s^{-1})(\cdot) \circ \varphi_s$ denotes the pull back of multivector fields along the diffeomorphism φ_s.

Consequently

$$\frac{d}{dt}|_{t=s}\varphi_t^*(\Pi) = \varphi_s^*([X_s,\Pi]_{SN}) = 0$$

since X_s is Poisson and hence

$$\begin{aligned} \varphi_s^*(\{f,g\}_\Pi) &= -\varphi_s^*([[\Pi,f]_{SN},g]_{SN}) = -[[\varphi_s^*(\Pi),\varphi_s^*(f)]_{SN},\varphi^*(g)]_{SN} \\ &= -[[\Pi,\varphi_s^*(f)]_{SN},\varphi_s^*(g)]_{SN} = \{\varphi_s^*(f),\varphi_s^*(g)\}_\Pi. \end{aligned}$$

□

COROLLARY 1.14. *Every smooth one-parameter family of diffeomorphisms generated by a smooth one-parameter family of Hamiltonian vector fields is a smooth one-parameter family of Poisson diffeomorphisms.*

DEFINITION 1.15. Let (M,Π) be a Poisson manifold. A smooth one-parameter family of diffeomorphisms $(\varphi_t)_{t\in[0,1]}$ starting at the identity is a *smooth one-parameter family of Hamiltonian diffeomorphisms* if a smooth function $F : M \times [0,1] \to \mathbb{R}$ exists such that

$$\frac{d}{dt}|_{t=s}\varphi_t = X_{F_s}|_{\varphi_s}$$

holds for arbitrary $t \in [0,1]$. Here $F_s := F(\cdot,s)$ denotes the restriction of F to $M \times \{s\}$. We denote the *set of smooth one-parameter families of Hamiltonian diffeomorphisms* of (M,Π) by $\underline{\mathrm{Ham}}(M,\Pi)$.

We say that a diffeomorphisms φ is a *Hamiltonian diffeomorphism* if a smooth one-parameter family of Hamiltonian diffeomorphisms $(\varphi_t)_{t\in[0,1]}$ exists such that $\varphi_1 = \varphi$. We denote the *set of Hamiltonian diffeomorphisms* of (M,Π) by $\mathrm{Ham}(M,\Pi)$.

Lemma 1.16. *Given a Poisson manifold* (M,Π)*, the composition of diffeomorphisms equips the sets* $\underline{\mathrm{Ham}}(E,\Pi)$ *and* $\mathrm{Ham}(M,\Pi)$ *with the structure of groups.*

We refer to $\underline{\mathrm{Ham}}(M,\Pi)$ *as the* group of smooth one-parameter families of Hamiltonian diffeomorphisms *of* (M,Π) *and to* $\mathrm{Ham}(M,\Pi)$ *as the* group of Hamiltonian diffeomorphisms *of* (M,Π) *respectively.*

Proof. See Lemma 5, Chapter 6. □

Lemma 1.17. *Let* (M,Π) *be a Poisson manifold. The following sequence is a complex of* $\mathbb{R}$*-modules*

$$0 \longrightarrow \mathcal{C}^\infty(M) \xrightarrow{[\Pi,\cdot]_{SN}} \Gamma(TM) \xrightarrow{[\Pi,\cdot]_{SN}} \Gamma(\wedge^2 TM) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Gamma(\wedge^n TM) \xrightarrow{[\Pi,\cdot]_{SN}} \Gamma(\wedge^{(n+1)} TM) \longrightarrow \cdots .$$

Moreover $[\Pi,\cdot]_{SN}$ *is a graded derivation of* $\mathcal{V}^\bullet(M)$ *of degree* $+1$*, i.e.* $[\Pi,\cdot]_{SN}$ *is a differential.*

Proof. The graded Jacobi identity for $[\cdot,\cdot]_{NS}$ implies

$$[\Pi,[\Pi,Y]_{SN}]_{SN} = [[\Pi,\Pi]_{SN},Y]_{SN} - [\Pi,[\Pi,Y]_{SN}]_{SN}$$

and hence $[\Pi,[\Pi,\cdot]_{SN}]_{SN} = 0$.

The derivation property follows from the fact that $[\cdot,\cdot]_{SN}$ is a graded biderivation, see Lemma 4.14 in Chapter 2. □

Definition 1.18. Given a Poisson manifold (M,Π), we refer to the complex

$$(\mathcal{V}(M), d_\Pi := [\Pi,\cdot]_{SN})$$

as the *Poisson complex* associated to (M,Π).

The cohomology $H_\Pi(M,\mathbb{R})$ is referred to as the *Poisson cohomology* of (M,Π).

Remark 1.19. This complex and its cohomology were first considered by Lichnerowicz ([**L**]).

The complex $(\mathcal{V}(M), d_\Pi)$ can be interpreted as the smooth graded manifold $T^*[1]M$ equipped with a cohomologcial vector field. As such it is equivalent to a Lie algebroid structure on T^*M, see [**dSW**] for instance.

Lemma 1.20. *Let* (M,Π) *be a Poisson manifold. Then the following diagram of* $\mathbb{R}$*-modules is commutative*

$$\begin{tikzcd}
\Omega^{n+1}(M) \arrow[r, "\wedge^{(n+1)}(-\Pi^\#)"] & \mathcal{V}^{n+1}(M) \\
\Omega^n(M) \arrow[u, "d_{DR}"] \arrow[r, "\wedge^n(-\Pi^\#)"] & \mathcal{V}^n(M) \arrow[u, "d_\Pi"']
\end{tikzcd}$$

and one obtains a morphism of cohomologies

$$[\wedge\Pi^{\#}] : H(M,\mathbb{R}) \to H_\Pi(M,\mathbb{R}).$$

Proof. Observe that both $d_\Pi \circ \wedge^n(-\Pi^{\#})$ and $\wedge^{(n+1)}(-\Pi^{\#}) \circ d_{DR}$ are graded derivations of $\Omega(M)$ with values in $\mathcal{V}(M)$ where $\mathcal{V}(M)$ is a module over $\Omega(M)$ via $ad \circ \wedge(-\Pi^{\#})$. Hence it suffices to know their images on functions and exact one-forms since they generate $\Omega(M)$ as an algebra.

So given any function f on M we compute

$$d_\Pi(f) = [\Pi, f]_{SN} = -(-[\Pi, f]_{SN}) = -\Pi^{\#}(d_{DR}f).$$

Furthermore

$$\begin{aligned}&\wedge^2(-\Pi^{\#})(d_{DR}(d_{DR}f)) = 0 \quad \text{and}\\ &d_\Pi(-\Pi^{\#}(d_{DR}f)) = [\Pi, [\Pi, f]_{SN}]_{SN} = \frac{1}{2}[[\Pi,\Pi]_{SN}, f]_{SN} = 0.\end{aligned}$$

□

Remark 1.21. The first four cohomology groups $H^0_\Pi(M,\mathbb{R})$,$H^1_\Pi(M,\mathbb{R})$, $H^2_\Pi(M,\mathbb{R})$ and $H^3_\Pi(M,\mathbb{R})$ posses a geometric interpretation. They contain Casimir functions, equivalence classes of Poisson vector fields modulo Hamiltonian vector fields, equivalence classes of infinitesimal deformations of Π modulo trivial deformations and obstructions to extending infinitesimal obstructions to formal ones, respectively. See [**dSW**] for more details.

Usually it is very hard to compute the Poisson cohomology for a given Poisson manifold. Let us add some remarks on special cases where the Poisson cohomology can be computed:

(a) For $(M, 0)$ the differential d_Π vanishes and consequently $H_0(M,\mathbb{R}) = \Gamma(\wedge TM)$.

(b) Consider the natural Poisson structure on the dual of a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. The space of smooth k-vector fields on $\mathfrak{g}^*$ is given by

$$\mathcal{C}^\infty(\mathfrak{g}^*) \otimes \wedge^k\mathfrak{g}^* \cong \mathrm{Hom}(\wedge^k\mathfrak{g}, \mathcal{C}^\infty(\mathfrak{g}^*)).$$

Under this identification we obtain a complex which is isomorphic to the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathcal{C}^\infty(\mathfrak{g}^*)$. Here the module structure is induced via pull back from the coadjoint action. Consequently,

$$H_\Pi(\mathfrak{g}^*,\mathbb{R}) \cong H(\mathfrak{g}, \mathcal{C}^\infty(\mathfrak{g}^*)).$$

Observe that if one restricts attention to the space of polynomial multivector fields on $\mathfrak{g}^*$ one obtains a subcomplex whose cohomology is isomorphic to $H(\mathfrak{g}, \mathcal{S}(\mathfrak{g}))$. Hence there is an inclusion of $H(\mathfrak{g}, \mathcal{S}(\mathfrak{g}))$ into the Poisson cohomology of $\mathfrak{g}^*$.

(c) The Poisson cohomology for quadratic Poisson bivector fields in $\mathbb{R}^2$ was computed in by Nakanishi in [**N**].

(d) For $\mathbb{R}^3$ equipped with the Poisson bivector field

$$\Pi_f := \frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right) + \frac{\partial f}{\partial z}\left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}\right)$$

for a function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying certain algebraic conditions, the Poisson cohomology was computed in [**P**].

(e) For (M, ω) symplectic the map $(\omega^{-1})^{\#}$ yields an isomorphism between $(\Omega(M), d_{DR})$ and $(\mathcal{V}(M), d_\Pi)$, hence $H_\Pi(M, \mathbb{R}) \cong H(M, \mathbb{R})$.

(f) The Poisson cohomology of (S^2, Π) where S^2 is interpreted as the homogeneous space $SU(2)/U(1)$ and $SU(2)$ is equipped with its standard Lie-Poisson bivector field was investigated in [**Gi**] and [**Roy2**].

Instead of trying to the compute the whole cohomology group $H_\Pi(M, \mathbb{R})$ one could also try to find special cohomology classes. Because $[\Pi, \Pi]_{SN} = 0$, the Poisson bivector field Π itself defines a cohomology class $[\Pi] \in H^2_\Pi(M, \mathbb{R})$, the *fundamental class* of (M, Π). Another cohomology class with geometric meaning is the *modular class* which was introduced by Weinstein ([**W4**]). Other characteristic classes whose geometric interpretation is not evident were constructed by Fernandes ([**Fe**]).

2. Coisotropic Submanifolds

Remark 2.1. Let S be a submanifold of the manifold M. The *conormal bundle* N^*S of S in M is defined by the following short exact sequence of vector bundles over S

$$0 \longrightarrow N^*S \longrightarrow T^*|_S M \longrightarrow T^*S \longrightarrow 0.$$

More explicitly $N^*_x S = \{\xi \in T^*_x M : \forall v \in T_x S : \xi(v) = 0\}$, i.e. N^*S is the annihilator of TS in $T|_S M$.

Definition 2.2. A submanifold S of a Poisson manifold (M, Π) is called *coisotropic* if the restriction of the vector bundle map

$$\Pi^{\#} : T^*M \to TM$$

to N^*S has image in TS.

Lemma 2.3. *Let S be a submanifold of the Poisson manifold (M, Π). It is a coisotropic submanifold of (M, Π) if and only of its* vanishing ideal

$$\mathcal{I}_S := \{f \in \mathcal{C}^\infty(M) : f|_S = 0\}$$

is a Lie subalgebra of the Lie algebra $(\mathcal{C}^\infty(M), \{\cdot, \cdot\}_\Pi)$.

REMARK 2.4. Lemma 2.3 can be found in [**W3**] for instance.

Observe that the property

$$\{\mathcal{I}_S, \mathcal{I}_S\}_\Pi \subset \mathcal{I}_S$$

can be checked locally, i.e. it is true if and only if it is true locally in submanifold charts of S in M.

PROOF. First let S be a coisotropic submanifold of (M, Π) and f, g two elements of its vanishing ideal. Consequently the restrictions of $d_{DR}f$ and $d_{DR}g$ to S yield sections of N^*S. Suppose $x \in S$. By Remark 1.8 we have

$$\{f, g\}(x) =< \Pi^\#(d_{DR}f)|_x, d_{DR}g|_x >$$

and since $\Pi^\#(d_{DR}f)|_x$ is an element of T_xS and $d_{DR}g|_x$ lies in the annihilator of T_xS in T_xM the term $< \Pi^\#(d_{DR}f)|_x, d_{DR}g|_x >$ vanishes. Hence $\{f, g\}_\Pi$ vanishes on S, i.e. it is an element of the vanishing ideal.

Now suppose that

$$\{\mathcal{I}_S, \mathcal{I}_S\}_\Pi \subset \mathcal{I}_S$$

is true. Any element $\xi \in N_x^*S$ can be written as $\xi = d_{DR}f|_x$ for some locally defined smooth function f vanishing on S. By definition

$$\Pi^\#(\xi) = X_f|_x = -[\Pi, f]_{SN}|_x$$

holds. If we apply $\Pi^\#(\xi)$ to an element λ of N_x^*S with $\lambda = d_{DR}g|_x$ for some locally defined smooth function g vanishing on S we obtain

$$< \Pi^\#(\xi), \lambda >=< \Pi^\#(d_{DR}f)|_x, d_{DR}g_x >= \{f, g\}_\Pi(x).$$

Because both f and g are locally defined functions lying in the vanishing ideal of S in M, so is $\{f, g\}_\Pi$. Hence $\Pi^\#(\xi)$ is annihilated by all elements of N_x^*S. The fact that the annihilator of the annihilator of some subvector space of a finite dimensional vector space is the subvector space itself implies that $\Pi^\#(\xi) \in T_xS$. $\square$

REMARK 2.5. A multiplicative ideal of a Poisson algebra that in addition is a Lie subalgebra is called a *coisotrope* of the Poisson algebra.

EXAMPLE 2.6.
(a) Every open subset of (M, Π) is a coisotropic submanifold regardless of Π, since N^*S is a rank 0 bundle.
(b) Every codimension one submanifold of (M, Π) is a coisotropic submanifold regardless of Π. This is due to the fact that the vanishing ideal of S in M is locally generated by one function and the Poisson bracket of a function with itself vanishes.
(c) Every submanifold of a Poisson manifold of the form $(M, 0)$ is a coisotropic submanifold.
(d) A point x of a Poisson manifold (M, Π) is a coisotropic submanifold if and only if the Poisson bivector field Π vanishes at x.

(e) A linear subspace of the dual $\mathfrak{g}^*$ of a finite dimensional real Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a coisotropic submanifold if and only if it is the annihilator of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This is easily verified with the help of a basis of $\mathfrak{g}$ that extends a basis of $\mathfrak{h}$.

(f) A submanifold S of a symplectic manifold (M, ω) is coisotropic if and only if the ω-orthogonal to TS given by

$$T_x^{\perp\omega} S := \{v \in T_x M : \forall u \in T_x S : \omega_x(v, u) = 0\}$$

is a subvector bundle of TS. In particular a submanifold S is *Lagrangian*, i.e. $T^{\perp\omega} S = TS$ holds, if and only if it is coisotropic and $\dim(S) = \frac{1}{2}\dim(M)$.

Lemma 2.7. *Let (M, Π) and (N, Λ) be two Poisson manifolds and $\varphi : M \to N$ a smooth map.*

The map φ is a Poisson map, *i.e.*

$$\varphi^*(\{f, g\}_\Lambda) = \{\varphi^*(f), \varphi^*(g)\}_\Pi$$

is satisfied for arbitrary $f, g \in \mathcal{C}^\infty(N)$, if and only if

$$\operatorname{graph}(\varphi) := \{(m, \varphi(m)) \in M \times N : m \in M\}$$

is a coisotropic submanifold of the Poisson manifold $(M \times N, -\Pi + \Lambda)$.

Proof. Denote the graph of φ by G. First we assume that G is a coisotropic submanifold of the Poisson manifold $(M \times N, -\Pi + \Lambda)$. Hence by definition the image of N^*G under

$$(-\Pi + \Lambda)^\# : T^*(M \times N) \to T(M \times N)$$

lies in TG. Since G is diffeomorphic to M, TG is diffeomorphic to TM. More explicitly, the bundle map

$$(\mathrm{id} + T\varphi) : TM \to TM \times TN, \quad v \mapsto v + T\varphi \cdot v$$

maps onto TG and induces the isomorphism $TM \cong TG$.

Consider the map $p : G \hookrightarrow M \times N \xrightarrow{\mathrm{pr}_2} N$ and pull back T^*N along p. There is a bundle map

$$(\varphi^* - \mathrm{id}) : p^*(T^*N) \to T^*M \times T^*N, \quad \xi \mapsto \varphi^*(\xi) - \xi.$$

It maps into the conormal bundle N^*G of G in $M \times N$ and for dimension reasons it actually induces an isomorphism $p^*(T^*N) \cong N^*G$. Now we can spell out the condition for G being coisotropic: it means that

$$(-\Pi + \Lambda)^\#(\varphi^*(\xi)|_x - \xi|_{\varphi(x)}) = -\Pi^\#(\varphi^*(\xi))|_x - \Lambda^\#(\xi)|_{\varphi(x)}$$

is equal to $v + T_x\varphi \cdot v$ for some $v \in T_x M$. This is equivalent to

$$\Lambda^\#(\xi)|_{\varphi(x)} = T_x\varphi \cdot \Pi^\#(\varphi^*(\xi))|_x$$

being satisfied for arbitrary $x \in M$ and ξ in $T^*_{\varphi(x)}N$. In particular

$$< \Lambda^{\#}(\xi), \nu > |_{\varphi(x)} =< \Pi^{\#}(\varphi^*(\xi))|_x, \varphi^*(\nu) >_x$$

holds for arbitrary $x \in M$ and $\xi, \nu \in T^*_{\varphi(x)}N$. Let f, g be two arbitrary smooth functions on N. We set $\xi = d_{DR}f$ and $\nu = d_{DR}g$ and by Remark 1.8 we obtain

$$\varphi^*(\{f,g\}_\Lambda)(x) = \{f,g\}_\Lambda|_{\varphi(x)} = \{\varphi^*(f), \varphi^*(g)\}_\Pi(x)$$

for arbitrary $x \in M$ and consequently φ is a Poisson map.

On the other hand suppose that $\varphi : M \to N$ is Poisson. Since any element of $T^*_{\varphi(x)}N$ can be written as the evaluation of an locally defined exact one-form at $\varphi(x)$, we recover

$$\Lambda^{\#}(\xi)|_{\varphi(x)} = T_x\varphi \cdot \Pi^{\#}(\varphi^*(\xi))|_x$$

from $\varphi^*(\{\cdot,\cdot\}_\Lambda) = \{\varphi^*(\cdot), \varphi^*(\cdot)\}_\Pi$ and hence the graph of φ is a coisotropic submanifold of $(M \times N, -\Pi + \Lambda)$. $\square$

REMARK 2.8. Lemma 2.7 is the starting point for an attempt to form a category whose objects are Poisson manifolds and the set of morphisms from (M,Π) to (N,Λ) is given by coisotropic submanifolds of $(M \times N, -\Pi+\Lambda)$ instead of Poisson maps from (M,Π) to (N,Λ). The only obstruction is that one has to impose certain transversality assumptions in order for the composition to be well-defined. See [**W3**] for more details on these matters.

LEMMA 2.9. *Let S be a coisotropic submanifold of a Poisson manifold (M,Π) and φ a Poisson diffeomorphism of (M,Π).*

Then the image of S under φ is a coisotropic submanifold of (M,Π).

PROOF. It is straightforward to check that the vanishing ideals of S and of $\varphi(S)$ in M are related by

$$(\varphi^{-1})^*(\mathcal{I}_S) = \mathcal{I}_{\varphi(S)}.$$

This relation and Lemma 2.3 imply

$$\begin{aligned} \{\mathcal{I}_{\varphi(S)}, \mathcal{I}_{\varphi(S)}\}_\Pi &= \{(\varphi^{-1})^*(\mathcal{I}_S), (\varphi^{-1})^*(\mathcal{I}_S)\}_\Pi \\ &= (\varphi^{-1})^*(\{\mathcal{I}_S, \mathcal{I}_S\}_\Pi) \subset (\varphi^{-1})^*(\mathcal{I}_S) = \mathcal{I}_{\varphi(S)}. \end{aligned}$$

Consequently $\mathcal{I}_{\varphi(S)}$ is a coisotrope of $(\mathcal{C}^\infty(M), \{\cdot,\cdot\}_\Pi)$ and by Lemma 2.3 one concludes that the submanifold $\varphi(S)$ is a coisotropic submanifold of (M,Π). $\square$

REMARK 2.10. Let S be a submanifold of a manifold M. The *normal bundle* of S in M is defined by the following short exact sequence of vector bundles over S

$$0 \longrightarrow TS \longrightarrow T|_S M \xrightarrow{\mathrm{pr}} NS \longrightarrow 0.$$

More explicitly, $N_xS = T_xM/T_xS$, i.e. NS is the quotient of $T|_SM$ by TS.

It is well-known that there always is an embedding of the manifold NS into M such that the restriction to S is the identity, see [**Hi**] for instance. Moreover any two such embeddings σ_0 and σ_1 can be connected by an isotopy of embeddings, i.e. there is a smooth map

$$\Sigma : NS \times [0,1] \to M$$

such that

(a) The two restrictions of Σ to $NS \times \{0\}$ and $NS \times \{1\}$ equal σ_0 and σ_1 respectively.
(b) The restriction of Σ to $S \times [0,1]$ is $(\mathrm{id}_S)_{t\in[0,1]}$.
(c) The map

$$\Sigma \times \mathrm{id} : NS \times [0,1] \to M \times [0,1], \quad (n,t) \mapsto (\Sigma(n,t),t)$$

is an embedding. Consequently the restriction of Σ to $NS \times \{t\}$ is an embedding for arbitrary $t \in [0,1]$.

The dual of the normal bundle NS is the conormal bundle N^*S introduced in Remark 2.1.

Definition 2.11. Given a coisotropic submanifold S of a Poisson manifold (M,Π), we define the *Lie algebroid differential*

$$\partial_\Pi : \Gamma(\wedge^\bullet NS) \to \Gamma(\wedge^{\bullet+1} NS)$$

of S in (M,Π) by

$$\partial_\Pi(X) := \wedge\,\mathrm{pr}\left([\Pi,\tilde{X}]_{SN}|_S\right)$$

where X denotes a section of $\wedge NS$ and $\tilde{X}$ denotes an arbitrary multivector field on an open neighbourhood of S in M such that the image of $\tilde{X}|_S \in \Gamma(\wedge T|_S M)$ under

$$\wedge pr : \Gamma(\wedge T|_S M) \to \Gamma(\wedge NS)$$

is equal to $X \in \Gamma(\wedge NS)$.

Lemma 2.12. *Let S be a coisotropic submanifold of a Poisson manifold (M,Π).*

(a) *The Lie algebroid differential ∂_Π is well-defined.*
(b) *It is a coboundary operator on $\Gamma(\wedge^\bullet NS)$, i.e. $\partial_\Pi \circ \partial_\Pi = 0$.*
(c) *It is a graded derivation of $\Gamma(\wedge^\bullet NS)$ of degree $+1$.*
(d) *The dual of the restriction of $-\Pi^\# : T^*M \to TM$ to N^*S induces a morphism of chain complexes from $(\Omega(S), d_{DR})$ to $(\Gamma(\wedge NS), \partial_\Pi)$. This morphism descends to a map from $H(S,\mathbb{R})$ to $H(\Gamma(\wedge NS), \partial_\Pi)$.*

Proof. (a) Two things have to be verified: every section X of $\wedge NS$ can be extended to a multivector field $\tilde{X}$ on an open neighbourhood of S in M and $\partial_\Pi(X)$ does not depend on the specific choice of the extension $\tilde{X}$.

Fix a Riemannian metric g on TM. This restricts to a fibre metric on $T|_S M$ and the orthogonal complement of TS in $T|_S M$ is isomorphic to NS, i.e. the short exact sequence defining NS splits and we obtain an injective vector bundle map

$$\wedge NS \hookrightarrow \wedge T|_S M.$$

This allows us to interpret X as a section of $\wedge T|_S M$. Next choose an embedding of NS into M such that the restriction to S is equal to the identity (see Remark 2.10). The image of this embedding is an open neighbourhood U of S in M which carries the structure of a vector bundle over S. The pull back of $T|_S U \to S$ along $U \to S$ is isomorphic to $TU \to U$. Consequently the following square is Cartesian

$$\begin{tikzcd} \wedge TU \arrow[r] \arrow[d] & \wedge T|_S U \arrow[d] \\ U \arrow[r] & S. \end{tikzcd}$$

The image of X under the pull back map $\Gamma(\wedge T|_S U) \to \Gamma(\wedge TU)$ is a multivector field $\tilde{X}$ on U such that $\wedge\operatorname{pr}(\tilde{X}|_S) = X$ holds.

The proof that $\partial_\Pi(X)$ is independent of the specific choice of extension $\tilde{X}$ reduces to checking that

$$\wedge\operatorname{pr}([\Pi, Z]_{SN}|_S) = 0 \tag{2.1}$$

holds for all multivector fields Z defined on an open neighbourhood of S in M such that $\wedge\operatorname{pr}(Z|_S) = 0$.

Observe that $\Gamma(\wedge TM)$ is locally generated by $\mathcal{C}^\infty(M)$ and $\Gamma(TM)$. Moreover $\wedge\operatorname{pr}(\cdot|_S) = 0$ and $\wedge\operatorname{pr}([\Pi, \cdot]_{SN}|_S) = 0$ are multiplicative conditions, i.e. if X and Y satisfy either of them, so does $X \wedge Y$. Assume the implication

$$\wedge\operatorname{pr}(Z|_S) = 0 \Rightarrow \wedge\operatorname{pr}([\Pi, Z]_{SN}|_S) = 0$$

holds for functions and vector fields. We saw that choosing a Riemannian metric on M and an embedding of NS into M as an open neighbourhood U of S in M leads to a splitting of TU into the direct sum of the pull back of TS and NS under $U \to S$ respectively. Now let V be a submanifold chart of S in U centered at $x \in S$. Choose a local frame $(\xi_1, \cdots, \xi_m)$ of TS and a local frame $(\chi_1, \ldots, \chi_n)$ of NS. These yield a local frame of TU. We assume that $\xi_r = \frac{\partial}{\partial x^r}$ for some local coordinate system $(x^1, \ldots, x^m)$ of S. Any section $X \in \Gamma(\wedge TV)$ is locally given by a sum of wedges of functions on U and pull backs of elements of the local frames of TS and NS respectively. Consider a term of the form

$$X = f\overline{\xi}_{i_1} \wedge \cdots \wedge \overline{\xi}_{i_k} \wedge \overline{\chi}_{j_i} \wedge \cdots \wedge \overline{\chi}_{j_l}$$

with $f \in \mathcal{C}^\infty(V)$ and "$-$" denotes the pull back. The condition

$$\wedge\operatorname{pr}(X|_S) = 0$$

implies that $f|_S = 0$ or $k \neq 0$ since $\mathrm{pr}(\xi_r) = 0$ and $\mathrm{pr}(\chi_s|_S) = \chi_s$ hold for arbitrary $r = 1, \ldots, m$ and $s = 1, \ldots, n$ respectively. By assumption either $\mathrm{pr}([\Pi, f]_{NS}|_S) = 0$ or $\wedge\,\mathrm{pr}([\Pi, \overline{\xi}_{i_1}]_{SN}|_S) = 0$. But if an element Y of $\Gamma(\wedge TV)$ satisfies $\wedge\,\mathrm{pr}(Y|_S) = 0$ and $\wedge\,\mathrm{pr}([\Pi, Y]_{SN}|_S) = 0$ then

$$\mathrm{pr}([\Pi, Y \wedge Z]_{SN}|_S) = 0$$

holds for arbitrary $Z \in \Gamma(\wedge TV)$. Consequently $\wedge\,\mathrm{pr}([\Pi, X]_{SN}|_S)$ vanishes for arbitrary $X \in \Gamma(\wedge TV)$ with $\wedge\,\mathrm{pr}(X|_S) = 0$.

It remains to show that $f|_S = 0$ implies $\mathrm{pr}([\Pi, f]_{SN}|_S) = 0$ for $f \in \mathcal{C}^\infty(M)$ and that for $X \in \Gamma(TM)$ with $\mathrm{pr}(X|_S) = 0$, $\wedge\,\mathrm{pr}([\Pi, X]_{SN}|_S) = 0$ holds. Pick a function f with $f|_S = 0$. Consequently $d_{DR}f|_S$ is a section of N^*S. Recall that

$$-[\Pi, f]_{SN}|_S = \Pi^\#(d_{DR}f)|_S.$$

Because S is a coisotropic submanifold of (M, Π), the image of $d_{DR}f|_S$ under $\Pi^\#|_S$ is a section of TS. Hence the image of $[\Pi, f]_{SN}|_S$ under $\mathrm{pr} : T|_S M \to NS = T|_S M/TS$ vanishes. On the other hand pick a local frame $(\overline{\xi}_1, \ldots, \overline{\xi}_m, \overline{\chi}_1, \ldots, \overline{\chi}_n)$ of TM as above. The elements $\overline{\xi}_r$ satisfy $P(\overline{\xi}_r|_S) = 0$. In this frame the Poisson bivector field is of the form

$$\frac{1}{2}\sum_{i,j=1}^{m} \Pi^{ij}\overline{\xi}_i \wedge \overline{\xi}_j + \sum_{i,\alpha} \Pi^{i\alpha}\overline{\xi}_i \wedge \overline{\chi}_\alpha + \frac{1}{2}\sum_{\alpha,\beta=1}^{n} \Pi^{\alpha\beta}\overline{\chi}_\alpha \wedge \overline{\chi}_\beta.$$

The fact that S is a coisotropic submanifold of (M, Π) is equivalent to $\Pi^{\alpha\beta}|_S = 0$ for arbitrary α, $\beta = 1, \ldots, n$. We obtain

$$\mathrm{pr}([\Pi, \overline{\xi}_r]_{SN}|_S) = -\frac{1}{2}\sum_{\alpha,\beta=1}^{n} \left(\frac{\partial \Pi^{\alpha\beta}}{\partial x^r}\right)|_S \chi_\alpha \wedge \chi_\beta.$$

Since $\Pi^{\alpha\beta}$ vanishes on S, so does $\frac{\partial \Pi^{\alpha\beta}}{\partial x^r}$ – recall that x^r is a local coordinate function on S.

(b) Given $X \in \Gamma(\wedge NS)$ we want to compute

$$\partial_\Pi(\partial_\Pi(X)) = \wedge\,\mathrm{pr}([\Pi, \wedge\,\mathrm{pr}(\widetilde{[\Pi, \tilde{X}]_{SN}|_S})]_{SN}|_S)$$

where $\tilde{X}$ is some extension of X to a multivector field on an open neighbourhood of S in M. We are free to choose any extension of $\wedge\,\mathrm{pr}([\Pi, \tilde{X}]_{SN}|_S)$ to a locally defined multivector field and we pick the extension

$$\wedge\,\mathrm{pr}(\widetilde{[\Pi, \tilde{X}]_{SN}|_S}) := [\Pi, \tilde{X}]_{SN}.$$

Consequently

$$\partial_\Pi(\partial_\Pi(X)) = \wedge\,\mathrm{pr}([\Pi, [\Pi, \tilde{X}]_{SN}]_{SN}|_S) = \wedge\,\mathrm{pr}(\frac{1}{2}[[\Pi, \Pi]_{SN}, \tilde{X}]_{SN}|_S) = 0.$$

(c) Given two homogeneous elements $X, Y \in \Gamma(\wedge NS)$ we want to compute

$$\partial_\Pi(X \wedge Y) = \wedge\,\mathrm{pr}([\Pi, \widetilde{X \wedge Y}]_{SN}|_S).$$

Choose two extensions $\tilde{X}$ and $\tilde{Y}$ of X and Y respectively. Then $\tilde{X} \wedge \tilde{Y}$ is an extension of $X \wedge Y$ and hence

$$\begin{aligned}
\partial_\Pi(X \wedge Y) &= \wedge\,\mathrm{pr}([\Pi, \tilde{X} \wedge \tilde{Y}]_{SN}|_S) \\
&= \wedge\,\mathrm{pr}\left(([\Pi, \tilde{X}]_{SN} \wedge \tilde{Y} + (-1)^{|X|}\tilde{X} \wedge [\Pi, \tilde{Y}]_{SN})|_S\right) \\
&= \wedge\,\mathrm{pr}([\Pi, \tilde{X}]_{SN}|_S) \wedge Y + (-1)^{|X|} X \wedge P([\Pi, \tilde{Y}]_{SN}|_S) \\
&= \partial_\Pi(X) \wedge Y + (-1)^{|X|} X \wedge \partial_\Pi(Y).
\end{aligned}$$

(d) We claim that the diagram

$$\begin{array}{ccc}
\Omega^{(n+1)}(S) & \xrightarrow{\wedge^{(n+1)}\zeta} & \Gamma(\wedge^{(n+1)} NS) \\
{\scriptstyle d_{DR}}\uparrow & & \uparrow{\scriptstyle \partial_\Pi} \\
\Omega^n(S) & \xrightarrow{\wedge^n\zeta} & \Gamma(\wedge^n NS)
\end{array}$$

commutes, where $\zeta : T^*S \to NS$ is the vector bundle map dual to $\Pi|_{N^*S} : N^*S \to TS$. Both $\wedge^{(n+1)}\zeta \circ d_{DR}$ and $\partial_\Pi \circ \wedge^n\zeta$ are graded derivations of $\Omega(S)$ with values in the $\Omega(S)$ module $(\Gamma(\wedge NS), ad \circ \wedge\zeta)$ degree $+1$. Since $\Omega(M)$ is locally generated by $\mathcal{C}^\infty(M)$ and exact one-forms it is enough to check the commutativity of the above diagrams on such elements.

Given $f \in \mathcal{C}^\infty(S)$, we have to compare the two sections

$$\zeta(d_{DR}f)(\cdot) =< d_{DR}f, \Pi^\#|_S(\cdot) > \quad \text{and}$$

$$\mathrm{pr}([\Pi, \tilde{f}]|_S)$$

of $\Gamma(NS)$. To this end we pick an arbitrary $\lambda \in \Gamma(N^*S)$ and compute

$$\begin{aligned}
< \mathrm{pr}([\Pi, \tilde{f}]|_S), \lambda > &= < [\Pi, \tilde{f}]_{SN}|_S, \lambda > \\
&= < -\Pi^\#(d_{DR}\tilde{f})|_S, \lambda > \\
&= < d_{DR}\tilde{f}|_S, \Pi^\#|_S(\lambda) > \\
&= < d_{DR}f, \Pi^\#|_S(\lambda) > \\
&= \zeta(d_{DR}f)(\lambda)
\end{aligned}$$

where we used the facts that Π is skew self-adjoint and that N^*S is the annihilator of TS in $T|_SM$.

Consider the exact one-form $d_{DR}f$. On the one hand we have

$$(\wedge\zeta)(d_{DR}(d_{DR}f)) = 0.$$

One the other hand we have show that

$$\partial_\Pi(\zeta(d_{DR}f)) = \wedge\,\mathrm{pr}([\Pi, \zeta\widetilde{(d_{DR}f)}]_{SN}|_S)$$

vanishes. We proved before that $\mathrm{pr}([\Pi,\tilde{f}]_{SN}|_S)=\zeta(d_{DR}f)$, hence

$$[\Pi,\tilde{f}]_{SN}$$

is an extension of $\zeta(d_{DR}f)$ and we obtain

$$\partial_\Pi(\zeta(d_{DR}f)) = \wedge\,\mathrm{pr}([\Pi,[\Pi,\tilde{f}]_{SN}]_{SN}|_S) = \wedge\,\mathrm{pr}(0) = 0.$$

□

Definition 2.13. Let S be a coisotropic submanifold of a Poisson manifold (M,Π). The *Lie algebroid complex* associated to S in (M,Π) is $(\Gamma(\wedge^\bullet NS),\partial_\Pi)$ and the *Lie algebroid cohomology* of S in (M,Π) is $H^\bullet(\Gamma(\wedge NS),\partial_\Pi)$.

Remark 2.14. The complex $(\Gamma(\wedge NS),\partial_\Pi)$ can be seen as the smooth graded manifold $N^*[1]S$ equipped with some cohomological vector field that encodes ∂_Π. As such it is equivalent to a Lie algebroid structure on N^*S. The Lie algebroid structure of a coisotropic submanifold of a Poisson manifold is spelled out in [**W3**] for instance.

The zero cohomology group $H^0(\Gamma(\wedge NS),\partial_\Pi)$ has a geometric interpretation which we explain next.

Definition 2.15. Let S be a coisotropic submanifold of a Poisson manifold (M,Π). The image of

$$\Pi^\#|_{N^*S} : N^*S \to TS$$

defines a distribution $\mathcal{F}(S)$ on S, i.e. one obtains a family of subvector spaces $\mathcal{F}_x(S)$ of T_xS parametrized by $x\in S$.

Definition 2.16. A distribution $\mathcal{F}$ of a smooth manifold M is called

(a) *smooth* if for every point $x\in M$ the vector space $\mathcal{F}_x$ is spanned by the restriction to x of locally defined vector fields on M which take values in $\mathcal{F}$;
(b) *involutive* if there exists a set $\mathcal{V}$ of locally defined vector fields on M that take values in $\mathcal{F}$ such that
  (i) for every $x\in M$ the vector space $\mathcal{F}_x$ is the linear span of
$$\{X|_x : X\in\mathcal{V}\}\quad\text{and}$$
  (ii) $[\mathcal{V},\mathcal{V}]_{SN}\subset\mathcal{V}$
  hold;
(c) *integrable* if for every $x\in M$ there is an immersed submanifold $\tilde{L}_x$ of M containing x such that for all $y\in\tilde{L}_x$ we have $T_yL_x=\mathcal{F}_y$.

Theorem 2.17. *Every smooth involutive distribution is integrable.*

Remark 2.18. This Theorem is usually attributed to Frobenius who established it in the regular case. It was later extended by Stefan ([**Ste**]) and Sussmann ([**Su**]) to the setting of Theorem 2.17. Our main reference is [**Mi**].

If a distribution is integrable there is a unique *maximal immersed submanifold* L_x associated to every point $x \in M$ such that for all $y \in L_x$ we have $T_y L_x = \mathcal{F}_y$. To be more precise, the set of all immersed submanifolds N satisfying

(a) $x \in N$
(b) $\forall y \in N$: $T_y N = \mathcal{F}_y$
(c) N is connected

is partially ordered with respect to inclusion and it contains a unique maximal element with respect to this partial ordering. This immersed submanifold is called the *leaf* through x.

Lemma 2.19. *The distribution $\mathcal{F}(S)$ associated to a coisotropic submanifold S of a Poisson manifold (M, Π) is smooth and involutive and hence integrable.*

Proof. Define $\mathcal{V}$ to be the set of locally defined vector fields given by $\Pi^{\#}(d_{DR} f)$ with f an arbitrary locally defined smooth function on M which vanishes on S. For arbitrary $x \in S$ every element ξ of the vector space $N^*_x S$ can be written as

$$\xi = (d_{DR} f)|_x$$

for some locally defined smooth function f vanishing on S. Consequently every vector v in $\mathcal{F}_x(S) = \Pi^{\#}(N^*_x)$ is given by

$$v = \Pi^{\#}(d_{DR} f)|_x$$

for some locally defined smooth function f vanishing on S. Hence the subset $\mathcal{V}$ of locally defined vector fields spans $\mathcal{F}_x(S)$ for arbitrary $x \in S$, i.e. the distribution $\mathcal{F}(S)$ is smooth.

We claim that $X, Y \in \mathcal{V}$ implies $[X, Y]_{SN} \in \mathcal{V}$. Suppose $X = \Pi^{\#}(d_{DR} f)$ and $Y = \Pi^{\#}(d_{DR} g)$ for two locally defined smooth functions f and g that vanish on S. Since f and g are locally defined elements of the vanishing ideal of S in M, so is $\{f, g\}_\Pi$ and because of

$$\begin{aligned}
[\Pi^{\#}(d_{DR} f), \Pi^{\#}(d_{DR} g)]_{SN} &= [-[\Pi, f]_{SN}, -[\Pi, g]_{SN}]_{SN} \\
&= [\Pi, [f, [\Pi, g]_{SN}]_{SN}]_{SN} + [f, [\Pi, [\Pi, g]_{SN}]_{SN}]_{SN} \\
&= [\Pi, -[[\Pi, g]_{SN}, f]_{SN}]_{SN} + [f, 0]_{SN} \\
&= [\Pi, \{g, f\}_\Pi]_{SN} \\
&= -[\Pi, \{f, g\}_\Pi]_{SN} \\
&= \Pi^{\#}(d_{DR}\{f, g\}_\Pi)
\end{aligned}$$

the set $\mathcal{V}$ is closed under $[\cdot, \cdot]_{SN}$. This means that the foliation $\mathcal{F}(S)$ is involutive. $\square$

Remark 2.20. By Lemma 2.19 the distribution $\mathcal{F}(S)$ associated to a coisotropic submanifold S of a Poisson manifold (M, Π) is integrable, hence there is a unique leaf through every point $x \in S$. We denote the set of leaves by $\underline{S}$ and refer to it as the *quotient space*. It is the set of equivalence classes of elements of S under the equivalence relation $\sim_{\mathcal{F}(S)}$ defined by

$$x \sim_{\mathcal{F}(S)} y :\Leftrightarrow x \text{ and } y \text{ lie in the same leaf.}$$

We equip the set $\underline{S}$ with the final topology with respect to the natural projection $S \to \underline{S}$, i.e. the open subsets of $\underline{S}$ are those subsets whose preimages in S are open.

The quotient space $\underline{S}$ is usually a very badly behaved space, e.g. it might be non-Hausdorff. In case it is Hausdorff one would like to equip the topological space $\underline{S}$ with the structure of a smooth manifold such that $S \to \underline{S}$ becomes submersive. However even if $\underline{S}$ is Hausdorff such a smooth structure on $\underline{S}$ might not exist.

Definition 2.21. Let S be a coisotropic submanifold of a Poisson manifold (M, Π).

The *quotient algebra* $\mathcal{A}(\underline{S})$ is the set of all elements of $\mathcal{C}^\infty(S)$ that are annihilated by all local vector fields on S with values in $\mathcal{F}(S)$.

Lemma 2.22. *The quotient algebra $\mathcal{A}(\underline{S})$ associated to a coisotropic submanifold S of a Poisson manifold (M, Π) is a subalgebra of $\mathcal{C}^\infty(S)$. Furthermore the map*

$$\mathcal{A}(\underline{S}) \times \mathcal{A}(\underline{S}) \to \mathcal{A}(\underline{S}), \quad (f, g) \mapsto \{\tilde{f}, \tilde{g}\}_\Pi|_S$$

where $\tilde{f}$ and $\tilde{g}$ are two locally defined smooth functions on M whose restrictions to S are equal to f and g respectively, defines a Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$ on $\mathcal{A}(\underline{S})$.

Proof. First we proof that

$$\{h, k\}_\Pi|_S = 0$$

holds under the assumptions that h vanishes on S and that $k|_S$ is contained in $\mathcal{A}(\underline{S})$. Recall that

$$\{h, k\}_\Pi =< \Pi^\#(d_{DR}h), d_{DR}k >= \left(\Pi^\#(d_{DR}h)\right)(k)$$

holds. Because h vanishes on S, the one-form $d_{DR}h$ is a locally defined section of N^*S. The locally defined vector field $\Pi^\#(d_{DR}h)$ takes values in the distribution $\mathcal{F}(S)$. But elements of $\mathcal{A}(\underline{S})$ are annihilated by such local vector fields. This implies that $\{\cdot,\cdot\}_{\underline{\Pi}}$ is well-defined.

All the other properties follow easily now: it is obvious that $\{\cdot,\cdot\}_{\underline{\Pi}}$ is skew-symmetric, and for arbitrary functions f, g and h in $\mathcal{A}(\underline{S})$ with extensions $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ respectively one computes

$$\begin{aligned} \{f, g \cdot h\}_{\underline{\Pi}} &= \{\tilde{f}, \widetilde{g \cdot h}\}_\Pi|_S = \{\tilde{f}, \tilde{g} \cdot \tilde{h}\}_\Pi|_S \\ &= \left(\{\tilde{f}, \tilde{g}\}_\Pi|_S\right) h + g \left(\{\tilde{f}, \tilde{h}\}_\Pi|_S\right) \\ &= \{f, g\}_{\underline{\Pi}} h + g\{f, h\}_{\underline{\Pi}} \end{aligned}$$

and

$$\{f,\{g,h\}_{\underline{\Pi}}\}_{\underline{\Pi}} = \{\tilde{f},\widetilde{\{\tilde{g},\tilde{h}\}_{\Pi}|_S}\}_{\Pi}|_S = \{\tilde{f},\{\tilde{g},\tilde{h}\}_{\Pi}\}_{\Pi}|_S.$$

These calculations imply that $\{\cdot,\cdot\}_{\underline{\Pi}}$ is a biderivation and a Lie bracket on $\mathcal{A}(\underline{S})$. ☐

REMARK 2.23. Let S be a coisotropic submanifold of a Poisson manifold (M,Π). Suppose that the quotient space $\underline{S}$ is Hausdorff and can be equipped with the structure of a smooth manifold such that the surjection $S \to \underline{S}$ is submersive. Observe that this condition determines the smooth structure on $\underline{S}$ completely: a real-valued function on $\underline{S}$ is smooth if and only if its composition with $S \to \underline{S}$ is smooth.

We claim that in this case the algebra of smooth functions $\mathcal{C}^\infty(\underline{S})$ is isomorphic to the algebra $\mathcal{A}(S)$. In fact, the algebra of smooth functions on $\underline{S}$ is isomorphic to the algebra of smooth functions on S that are constant along the leaves of $\mathcal{F}$. By definition the leaves of $\mathcal{F}$ are connected, thus the condition on functions to be constant along a leaf is equivalent to the condition that all local vector fields tangent to the leaf annihilate the functions under consideration. But this is exactly the condition we imposed on function of $\mathcal{C}^\infty(S)$ in order to lie in $\mathcal{A}(\underline{S})$. Because $\mathcal{C}^\infty(\underline{S})$ is isomorphic to $\mathcal{A}(\underline{S})$ as an algebra,it is also naturally equipped with a Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$, i.e. $\underline{S}$ comes along with a Poisson bivector field which we denote by $\underline{\Pi}$.

In summary $\mathcal{A}(\underline{S})$ seems to be a good candidate for the "algebra of smooth functions" on the quotient space $\underline{S}$, even in case S is lacking good topological or differential geometric properties.

LEMMA 2.24. *Let S be a coisotropic submanifold of a Poisson manifold* (M,Π).

Then the quotient algebra $\mathcal{A}(\underline{S})$ is isomorphic to the zero'th Lie algebroid cohomology $H^0(\Gamma(\wedge NS),\partial_\Pi)$ of S in (M,Π).

PROOF. The zero'th cohomology of the Lie algebroid complex of S in (M,Π) is the kernel of

$$\mathcal{C}^\infty(S) \xrightarrow{\partial_\Pi} \Gamma(NS)$$

i.e. a function f is in $H^0(\Gamma(\wedge NS),\partial_\Pi)$ if and only if

$$\partial_\Pi(f) = \mathrm{pr}([\Pi,\tilde{f}]_{SN}|_S)$$

vanishes. In the proof of part (d) of Lemma 2.12 the identity

$$< \mathrm{pr}([\Pi,\tilde{f}]_{SN}|_S),\lambda > = < \Pi^\#|_S(\lambda), d_{DR}f > = \left(\Pi^\#|_S(\lambda)\right)(f)$$

was established for arbitrary $\lambda \in \Gamma(N^*S)$. Hence $\partial_\Pi(f) = 0$ is equivalent to the condition that f is annihilated by all local vector fields on S with values in the distribution $\mathcal{F}(S)$. This holds if and only if $f \in \mathcal{A}(\underline{S})$. ☐

EXAMPLE 2.25. Similar to the computation of the Poisson cohomology of a given Poisson manifold, the computation of the Lie algebroid cohomology of a coisotropic submanifold is in general an unmanageable task. However there are special cases where information about the Lie algebroid complex and the cohomology beyond degree 0 is available.

(a) For an open submanifold S of (M, Π) the Lie algebroid complex is just $0 \to \mathcal{C}^\infty(S) \to 0$.
(b) Let S be a submanifold of (M, Π) of codimension 1. The Lie algebroid complex is

$$0 \to \mathcal{C}^\infty(S) \xrightarrow{\partial_\Pi} \Gamma(NS) \to 0$$

where $\partial_\Pi(f) = -\operatorname{pr}\left(X_{\tilde{f}}|_S\right)$. Recall that $\tilde{f}$ is an arbitrary extension of f to an open neighbourhood of S in M and that pr denotes the projection $T|_S M \to NS$. Consequently the first Lie algebroid cohomology of S is given by

$$\Gamma(NS)/\operatorname{pr}(\operatorname{Ham}(M)|_S),$$

i.e. the quotient of $\Gamma(NS)$ by the normal part of the restriction to S of all possible Hamiltonian vector fields on M.
(c) For any submanifold of the Poisson manifold $(M, 0)$ the differential ∂_Π vanishes and the Lie algebroid cohomology is just the space of sections of the exterior algebra of its normal bundle.
(d) Let $x \in M$ be a point of M where Π_x vanishes. Choosing a suitable chart arount x this amounts to considering a Poisson structure Π on $\mathbb{R}^n$ that vanishes at 0. Now Π can be interpreted as a map $\Pi^{i,j}(x)$ from $\mathbb{R}^n$ to $\mathfrak{o}(n)$, i.e. as a smooth function on $\mathbb{R}^n$ which takes values in the vector space of skew-symmetric $n \times n$-matrices. Because of $\Pi^{ij}(0) = 0$ there are smooth functions $\Pi^{ij}_k(x)$ such that

$$\Pi^{ij}(x) = \sum_{k=1}^{n} \Pi^{ij}_k(x) x^k.$$

Here x^k denotes the k'th linear coordinate on $\mathbb{R}^n$. It is straightforward to check that

$$\begin{array}{rcl} (\mathbb{R}^n)^* \times (\mathbb{R}^n)^* & \to & (\mathbb{R}^n)^* \\ (x^i, x^j) & \mapsto & \displaystyle\sum_{k=1}^{n} \Pi^{ij}_k(0) x^k \end{array}$$

defines the structure of a Lie algebra on $(\mathbb{R}^n)^*$. We denote this Lie algebra, which is known as the *linear approximation* of Π at x ([**W2**]), by $\mathfrak{g}_x$. Observe that this structure is independent of the specific choice of the functions $\Pi^{ij}_k(x)$.

The Lia algebroid complex associated to x can be identified with the Chevalley–Eilenberg complex of $\mathfrak{g}_x$ with values in the trivial module $\mathbb{R}$. This implies that the Lie algebroid cohomology of x is given by $H(\mathfrak{g}_x, \mathbb{R})$.

(e) Consider $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra of a finite dimensiona Lie algebra over $\mathbb{R}$. As observed in Examplex 2.6 (e), this implies that the annihilator $\mathfrak{h}^\circ$ of $\mathfrak{h}$ in $\mathfrak{g}$ is a coisotropic submanifold of $\mathfrak{g}^*$. The Lie algebroid complex of $\mathfrak{h}^0$ is given as follows: $\mathfrak{h}$ acts on $\mathfrak{g}$ by the adjoint action. Because $\mathfrak{h}$ is a Lie subalgebra one obtains an action on the quotient vector space $\mathfrak{g}/\mathfrak{h}$. This induces the structure of a module over $\mathfrak{h}$ on $\mathcal{C}^\infty\left((\mathfrak{g}/\mathfrak{h})^*\right)$. The Lie algebroid complex of $\mathfrak{h}^\circ$ coincides with the Lie algebra cohomology of $\mathfrak{h}$ with values in the module $\mathcal{C}^\infty\left((\mathfrak{g}/\mathfrak{h})^*\right)$. Consequently the Lie algebroid cohomology of $\mathfrak{h}^\circ$ is $H(\mathfrak{h}, \mathcal{C}^\infty\left((\mathfrak{g}/\mathfrak{h})^*\right))$.

(f) Assume S is a Lagrangian submanifold of a symplectic manifold. It is easy to check that in this case the morphism $(\Omega(L), d_{DR}) \to (\Gamma(\wedge NS), \partial_{\omega^{-1}})$ introduced in Lemma 2.12 (d) is an isomorphism of complexes and hence the Lie algebroid cohomology of L is isomorphic to its de Rham cohomology.

3. The homotopy Lie Algebroid

Remark 3.1. In the previous Section the Lie algebroid cohomology

$$(\Gamma(\wedge NS), \partial_\Pi)$$

associated to a coisotropic submanifold S of a Poisson manifold (M, Π) was introduced. We saw that the zero'th cohomology $H^0(\Gamma(\wedge NS), \partial_\Pi)$ is isomorphic to the quotient algebra $\mathcal{A}(\underline{S})$ which comes equipped with a Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$. Can this Poisson bracket be lifted from $H^0(\Gamma(\wedge NS), \partial_\Pi)$ to the cochain level, i.e. is there a natural structure on the complex $(\Gamma(\wedge NS), \partial_\Pi)$ that induces the Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$ on the zero'th cohomology? An affirmative answer to this question which will be presented below was found in [**OP**] and [**CF**].

Remark 3.2. Given a vector bundle $E \xrightarrow{p} S$ there is a short exact sequence

$$0 \longrightarrow TS \longrightarrow T|_S E \longrightarrow E \longrightarrow 0$$

of vector bundles over S. The inclusion $TS \to T|_S E$ is induced from the embedding of S into E as the zero section. The quotient of these two vector bundles is naturally isomorphic to the vector bundle E. Moreover this short exact sequence naturally splits: there is an inclusion of the vector bundle E into $T|_S E$ as the kernel of the vector bundle morphism $T|_S E \to TS$ induced by the projection $E \to S$. In fact the pull back $p^*(E) \to E$ of $E \to S$ along $E \xrightarrow{p} S$ is the kernel of $TE \to TS$. In particular $\Gamma(\wedge T|_S M)$ splits:

$$\Gamma(\wedge T|_S M) \cong \Gamma(\wedge E) \oplus \Gamma(\wedge^{\geq 1} TS \otimes \wedge E).$$

As a consequence of the previous paragraph we obtain an inclusion

$$I : \Gamma(\wedge E) \xrightarrow{p^*} \Gamma(\wedge p^*(E)) \hookrightarrow \Gamma(\wedge TE) = \mathcal{V}(E)$$

and a projection

$$P : \mathcal{V}(E) \to \Gamma(\wedge T|_S E) \to \Gamma(\wedge E).$$

Because of $P \circ I = \mathrm{id}$, the identity $(I \circ P) \circ (I \circ P) = (I \circ P)$ holds.

We claim that $I(\Gamma(\wedge E)[1])$ is an abelian Lie subalgebra of $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$. In fact, given $f \in \mathcal{C}^\infty(S)$ and $X \in \Gamma(E)$, consider

$$[I(X), I(f)]_{SN} = [I(X), p^*(f)]_{SN}.$$

By definition of $[\cdot,\cdot]_{SN}$ this is equal to the

$$\frac{d}{dt}|_{t=0}\left(p^*(f) \circ \phi_t^{I(X)}\right) = \frac{d}{dt}|_{t=0} f(p(\phi_t^{I(X)})) = \frac{d}{dt}|_{t=0}(f \circ p) = 0$$

where $\phi_t^{I(X)}$ is the one-parameter family of diffeomorphisms of E generated by $I(X)$. Observe that $\phi_t^{I(X)} : E \to E$ is given by a shift by $t \cdot X$ along the fibres. Similarly, given another section Y of E, we compute

$$[I(X), I(Y)]_{SN} = \frac{d}{dt}|_{t=0}(\phi_t^{I(X)})^*(I(Y)) = \frac{d}{dt}|_{t=0} I(Y)|_{\phi_t^{I(X)}} = \frac{d}{dt}|_{t=0} I(Y) = 0.$$

Observe that the algebra $I(\Gamma(\wedge E))$ is locally generated by $I(\mathcal{C}^\infty(S))$ and $I(\Gamma(E))$ and because $[\cdot,\cdot]_{SN}$ is a graded biderivation it suffices to check

$$[I(\Gamma(E)), I(\mathcal{C}^\infty(S))]_{SN} = 0 \quad \text{and} \quad [I(\Gamma(E)), I(\Gamma(E))]_{SN} = 0$$

in order to verify that $I(\Gamma(\wedge E))$ is an abelian Lie subalgebra of $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$.

Lemma 3.3. *Given a vector bundle $E \to S$, the graded vector space $\mathcal{V}(E)[1]$ splits into an abelian Lie subalgebra $I(\Gamma(\wedge E)[1])$ and a graded Lie subalgebra given by the kernel of*

$$P[1] : \mathcal{V}(E)[1] \to \Gamma(\wedge E)[1].$$

Consequently the quadruple

$$(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}, I(\Gamma(\wedge E)[1]), P[1])$$

is a V-algebra, see 3.1 in Chapter 2.

Proof. We have to prove that the kernel of P is closed under $[\cdot,\cdot]_{SN}$. Observe that the kernel consists of exactly those multivector fields on E whose restriction to S lies in $\Gamma(\wedge^{\geq 1} TS \otimes \wedge E)$. Let X and Y be two such multivector fields.

Choose a connection ∇ on $E \to S$. The horizontal lift of multivector fields uniquely extends to a morphism of algebras

$$i : \Gamma(\wedge T|_S N) \to \mathcal{V}(E)$$

by declaring i to be the vertical lift on sections of $\wedge E$. Let $\overline{X}$ and $\overline{Y}$ be the images of $X|_S$ and $Y|_S$ under i. Both $\overline{X} - X$ and $\overline{Y} - Y$ vanish when restricted to S.

Let V be a vector field on E whose restriction to S lies in $\Gamma(TS)$ and W a multivector field whose restriction to S vanishes. By definition

$$[V,W]_{SN} := \frac{d}{dt}|_{t=0}\left(\wedge(\phi_t^V)^*(W)\right)$$

holds and since the flow ϕ_t^V generated by V maps S to itself and W vanishes on S, so does $[V,W]_{SN}$. This implies that if U is a multivector field whose restriction to S lies in $\Gamma(\wedge^{\geq 1}TS \otimes \wedge NS)$ and W a multivector field whose restriction to S vanishes, $P([U,W]_{SN}) = 0$ holds.

Consequently we have

$$\begin{aligned} P([X,Y]_{SN}) &= P([\overline{X},Y]_{SN}) + P([X-\overline{X},Y]_{SN}) \\ &= P([\overline{X},\overline{Y}]_{SN}) + P(\overline{X},Y-\overline{Y}]_{SN}) \\ &= P([\overline{X},\overline{Y}]_{SN}). \end{aligned}$$

Suppose V and W are vector fields on S and $\overline{V}$ and $\overline{W}$ their horizontal lifts with respect to some connection ∇. Then the difference of $[\overline{V},\overline{W}]_{SN}$ and the horizontal lift of $[V,W]_{SN}$ is given by the contraction of the curvature R_∇ with V and W. However $R_\nabla(V,W)$ can be interpreted as a vertically acting vector field on E that vanishes when restricted to S. This implies

$$P([\overline{X},\overline{Y}]_{SN}) = 0.$$

An alternative proof that the kernel of P is closed under $[\cdot,\cdot]_{SN}$ can be found in [**BGHHW**]. □

Corollary 3.4. *Given a vector bundle $E \to S$ equipped with a Poisson bivector field Π, the higher derived brackets*

$$\begin{aligned} \tilde{\lambda}_n : S^n(\Gamma(\wedge E)[1]) &\to \Gamma(\wedge E)[2] \\ \xi_1 \otimes \cdots \otimes \xi_n &\mapsto P\left([[\cdots[[\Pi, I(\xi_1)]_{SN}, I(\xi_2)]_{SN}\cdots], I(\xi_n)]_{SN}\right) \end{aligned}$$

define the structure of an $L_\infty[1]$-algebra on $\Gamma(\wedge E)[1]$. This is equivalent to the structure of an L_∞-algebra on $\Gamma(\wedge E)$. We denote the associated family of structure maps by

$$(\lambda_n : \wedge^n(\Gamma(\wedge E)) \to \Gamma(\wedge E)[2-n])_{n\in\mathbb{N}}.$$

Proof. This is an immediate consequence of Theorem 3.5 in Chapter 2 and Lemma 3.3. □

Remark 3.5. Let S be a submanifold of a Poisson manifold (M,Π). Every embedding σ of the normal bundle NS of S in M into M whose restriction to S is the identity yields a vector bundle equipped with a Poisson bivector as follows: The image of σ is an open neighbourhood U of S in M and as such it inherits a Poisson bivector field $\Pi|_U$. The embedding σ induces an diffeomorphism $NS \cong U$ and the pull back of Π_U equips NS with a Poisson bivector field Π_σ.

Suppose S is a coisotropic submanifold of (M,Π). Then S is also a coisotropic submanifold of $(U,\Pi|_U)$ and consequently it is a coisotropic submanifold of the Poisson manifold (NS,Π_σ).

Definition 3.6. Let S be a submanifold of a Poisson manifold (M,Π) and σ an embedding of the normal bundle NS of S into M into M whose restriction to S is the identity.

The *homotopy Lie algebroid* of (S,σ) in (M,Π) is the graded algebra $\Gamma(\wedge NS)$ equipped with the L_∞-algebra given by the higher derived brackets $(\lambda_n^\sigma)_{n\in\mathbb{N}}$ associated to the Poisson structure Π_σ on the vector bundle $NS \to S$.

Lemma 3.7. *The homotopy Lie algebroid of (S,σ) in (M,Π) has the following properties:*

(a) *It is a P_∞-algebra structure – see Definition 4.15, Chapter 2 – i.e.*

$$\begin{aligned}&\lambda_n^\sigma(\chi_1\otimes\cdots\otimes\chi_{(n-1)}\otimes(\zeta\wedge\xi))\\ &\quad=\lambda_n^\sigma(\chi_1\otimes\cdots\otimes\chi_{n-1}\otimes\zeta)\wedge\xi+(-1)^{|\zeta||\xi|}\lambda_n^\sigma(\chi_1\otimes\cdots\otimes\chi_{n-1}\otimes\xi)\wedge\zeta\end{aligned}$$

holds for all homogeneous $\chi_1,\dots,\chi_{n-1},\zeta$ and ξ in $\Gamma(\wedge NS)$.

(b) *It is flat, i.e. the component $\lambda_0^\sigma\in\Gamma(\wedge^2E)$ vanishes, if and only if S is a coisotropic submanifold of (M,Π). In that case the following statements hold:*
   (i) *the component*
   $$\lambda_1^\sigma:\Gamma(\wedge NS)\to\Gamma(\wedge NS)[1]$$
   equals the Lie algebroid differential ∂_Π of S in (M,Π), see Definition 2.11,
   (ii) *the component*
   $$\lambda_2^\sigma:\Gamma(\wedge NS)\times\Gamma(\wedge NS)\to\Gamma(\wedge NS)$$
   induces a graded Poisson bracket $[\lambda_2^\sigma]$ on $H(\Gamma(\wedge NS),\partial_\Pi)$,
   (iii) *the graded Poisson bracket $[\lambda_2^\sigma]$ restricts to a Poisson bracket on $H^0(\Gamma(\wedge NS),\partial_\Pi)$ which is isomorphic to the Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$ introduced in Lemma 2.22 under the identification*
   $$\mathcal{A}(\underline{S})=H^0(\Gamma(\wedge NS),\partial_\Pi)$$
   which was established in Lemma 2.24.

Proof. (a) This follows easily from the fact that $[\cdot,\cdot]_{SN}$ is a graded biderivation.

(b) The zero'th component of the L_∞-algebra structure on $\Gamma(\wedge NS)$ is given by
$$\lambda_0^\sigma=P(\Pi_\sigma)$$

where P is evaluation on S followed by the projection $\wedge T|_S \to \wedge NS$. For ρ an arbitrary section of N^*S we calculate

$$< \lambda_0^\sigma, \rho > = < P(\Pi_\sigma), \rho > = \left(\Pi_\sigma^\#|_S(\rho)\right) \text{mod TS}.$$

Consequently $\lambda_0^\sigma = 0$ if and only if $\Pi_\sigma^\#|_S$ maps any section of N^*S to TS, i.e. if and only if S is a coisotropic submanifold of (NS, Π_σ) and equivalently a coisotropic submanifold of $(U, \Pi|_U)$ for $U := \sigma(NS)$. Being coisotropic is a local property, i.e. S is a coisotropic submanifold of $(U, \Pi|_U)$ if and only if it is a coisotropic submanifold of (M, Π).

From now on we will assume that S is a coisotropic submanifold of (NS, Π_σ). The Lie algebroid differential ∂_Π of S in (NS, Π_σ) was defined by

$$\partial_\Pi(X) := \wedge\, \mathrm{pr}\left([\Pi_\sigma, \tilde{X}]|_S\right)$$

where $\tilde{X}$ is some extension of X to a multivector field defined on an open neighbourhood of S in NS, and $\wedge\,\mathrm{pr}$ is the projection $T|_S M \to NS$. Observe that $I(X)$ is a possible extension of X and that $\wedge\,\mathrm{pr}(\cdot|_S)$ is equal to $P : \mathcal{V}(NS) \to \Gamma(\wedge NS)$. In summary we obtain

$$\partial_\Pi(X) := P([\Pi, I(X)]_{SN})$$

which is exactly the formula for $\tilde{\lambda}_1^\sigma$. Moreover the décalage-isomorphism does not change the sign of the differential.

The structure map λ_2^σ is a graded skew-symmetric map

$$\Gamma(\wedge NS) \times \Gamma(\wedge NS) \to \Gamma(\wedge NS)$$

of degree 0. The vanishing of λ_0^σ and of the second Jacobiator associated to $(\lambda_n^\sigma)_{n\in\mathbb{N}}$ implies that

$$\partial_\Pi\left(\lambda_2^\sigma(\zeta \otimes \xi)\right) = \pm\lambda_2^\sigma(\partial_\Pi(\zeta) \otimes \xi) \pm \lambda_2^\sigma(\zeta \otimes \partial_\Pi(\xi))$$

holds for arbitrary ζ, ξ in $\Gamma(\wedge NS)$. Assume ζ and ξ are two cocycles of $(\Gamma(\wedge NS), \partial_\Pi)$, i.e.

$$\partial_\Pi(\zeta) = 0 = \partial_\Pi(\xi)$$

and consequently

$$\partial_\Pi\left(\lambda^2(\zeta \otimes \xi)\right) = 0.$$

Hence we obtain a cohomology class $[\lambda^2(\zeta\otimes\xi)]$ in $H(\Gamma(\wedge NS), \partial_\Pi)$. Moreover if ζ is changed by a coboundary $\partial_\Pi(\mu)$ we obtain

$$\begin{aligned}
\lambda_2^\sigma((\zeta + \partial_\Pi(\mu)) \otimes \xi) &= \lambda_2^\sigma(\zeta \otimes \xi) + \lambda_2^\sigma(\partial_\Pi(\mu) \otimes \xi) \\
&= \lambda_2^\sigma(\zeta \otimes \xi) \pm \partial_\Pi\left(\lambda_2^\sigma(\mu \otimes \xi)\right)
\end{aligned}$$

and so the cohomology class $[\lambda_2^\sigma(\zeta + \partial_\Pi(\mu), \xi)]$ is equal to $[\lambda_2^\sigma(\zeta, \xi)]$. In summary λ_2^σ induces a bilinear skew-symmetric operation

$$\begin{aligned} [\lambda_2^\sigma] : H(\Gamma(\wedge NS), \partial_\Pi) \times H(\Gamma(\wedge NS), \partial_\Pi) &\to H(\Gamma(\wedge NS), \partial_\Pi) \\ ([\zeta], [\xi]) &\mapsto [\lambda_2^\sigma(\zeta \otimes \xi)]. \end{aligned}$$

The vanishing of λ_0^σ and of the third Jacobiator associated to $(\lambda_n^\sigma)_{n\in\mathbb{N}}$ implies

$$\begin{aligned} &\lambda_2^\sigma(\zeta \otimes \lambda_2^\sigma(\xi \otimes \mu)) + \lambda_2^\sigma(\lambda_2^\sigma(\zeta \otimes \xi) \otimes \mu) + (-1)^{|\zeta||\xi|}\lambda_2^\sigma(\xi \otimes \lambda_2^\sigma(\zeta \otimes \mu)) \\ &= \pm\partial_\Pi\left(\lambda_3^\sigma(\zeta \otimes \xi \otimes \mu)\right) \pm \lambda_3^\sigma(\partial_\Pi(\zeta) \otimes \xi \otimes \mu) \\ &\qquad \pm\lambda_3^\sigma(\zeta \otimes \partial_\Pi(\xi) \otimes \mu) \pm \lambda_3^\sigma(\zeta \otimes \xi \otimes \partial_\Pi(\mu)) \end{aligned}$$

where ζ, ξ and μ are arbitrary homogeneous sections of $\wedge NS$. Suppose these three elements represent cohomology classes in $H(\Gamma(\wedge NS), \partial_\Pi)$. Then the above identity implies that λ_2^σ satisfies the graded Jacobi identity up to a coboundary term. Consequently the graded Jacobi identity holds for the induced operation $[\lambda_2^\sigma]$ on cohomology.

Since ∂_Π is a graded derivation of degree 1, $H(\Gamma(\wedge NS), \partial_\Pi)$ inherits the structure of a graded commutative algebra from $\Gamma(\wedge NS)$. Now λ_2^σ is a graded biderivation for the algebra structure on $\Gamma(\wedge NS)$ and hence the operation $[\lambda_2^\sigma]$ is a biderivation for the induces algebra structure on $H(\Gamma(\wedge NS), \partial_\Pi)$.

Because the second structure map λ_2^σ is of total degree 0 its restriction to $H^0(\Gamma(\wedge NS), \partial_\Pi)$ takes values in $H^0(\Gamma(\wedge NS), \partial_\Pi)$. The graded skew symmetry, the graded Jacobi identity and the graded derivation property reduce to skew-symmetry, Jacobi identity and the usual derivation property on $H^0(\Gamma(\wedge NS), \partial_\Pi)$, i.e. $[\lambda_2^\sigma]$ equips $H^0(\Gamma(\wedge NS), \partial_\Pi)$ with the structure of a Poisson algebra.

Given two elements of $H^0(\Gamma(\wedge NS), \partial_\Pi)$, i.e. two functions f and g on S in the kernel of

$$\mathcal{C}^\infty(S) \xrightarrow{\partial_\Pi} \Gamma(NS).$$

The Poisson bracket $[\tilde{\lambda}_2^\sigma]$ is given by

$$[\tilde{\lambda}_2^\sigma](f \otimes g) = \tilde{\lambda}_2^\sigma(f \otimes g) = P\left([[\Pi, I(f)]_{SN}, I(g)]_{SN}\right).$$

Observe that $I(f)$ and $I(g)$ are extensions of f and g to functions on NS and that $P = \wedge\,\mathrm{pr}(\cdot|_S)$. This observation yields

$$[\tilde{\lambda}_2^\sigma](f \otimes g) = -\{f, g\}_{\underline{\Pi}}.$$

The décalage-isomorphism adds a minus sign when $\tilde{\lambda}_2^\sigma$ is translated into λ_2^σ.

□

We may summerize these results in case of a coisotropic submanifold as follows:

THEOREM 3.8. *Let S be a coisotropic submanifold of a Poisson manifold (M, Π). It is possible to find structure maps*

$$\lambda_n : \wedge^n(\Gamma(\wedge NS)) \to \Gamma(\wedge NS)[2-n]$$

for $n \geq 2$ such that

(a) *together with $\lambda_1 := \partial_\Pi$ the family of structure maps $(\lambda_n)_{n\geq 1}$ equips $\Gamma(\wedge NS)$ with the structure of an L_∞-algebra and*
(b) *the operation $[\lambda_2]$ on $H^0(\Gamma(\wedge NS), \partial_\Pi) \cong \mathcal{A}(\underline{S})$ induced by λ_2 is equal to $\{\cdot,\cdot\}_{\underline{\Pi}}$.*

REMARK 3.9. Theorem 3.8 was first proved in [**OP**] for S a coisotropic submanifold of a symplectic manifold. The construction was extended to a general Poisson manifold in [**CF**] where also arbitrary submanifolds S where taken into account.

In case S is a coisotropic submanifold of a Poisson manifold (M, Π) the L_∞-algebra structure $(\lambda_n^\sigma)_{n\geq 1}$ associated to S and an embedding $\sigma : NS \to M$ was baptized *strong homotopy Lie algebroid* associated to (S, σ) ([**OP**]). The adjective "strong" refers to the fact that the L_∞-algebra $(\lambda_n^\sigma)_{n\geq 1}$ is flat, i.e. the component $\lambda_0^\sigma \in \Gamma(\wedge^2 NS)$ vanishes. Since this usage of the adjective "strong" is in conflict with the usual meaning in the theory of higher homotopy structures where it is used to indicate the fact that homotopies at all levels are provided, we will not use the term strong homotopy Lie algebroid in the following.

In [**CF**] it was observed that for an arbitrary submanifold S and embedding σ the structure maps $(\lambda_n^\sigma)_{n\in\mathbb{N}}$ are all multiderivations with respect to $\wedge$ and the term P_∞*-algebras* was coined.

REMARK 3.10. Observe that the L_∞-algebra structure $(\lambda_n^\sigma)_{n\in\mathbb{N}}$ associated to a submanifold S of a Poisson manifold (M, Π) and an embedding $\sigma : NS \to M$ whose restriction to S is the identity only depends on the values of Π_σ in an arbitrarily small open neighbourhood of S in NS. In fact, the structure maps only depend on all the derivatives of Π in fibre directions at S. To be more precise, let $\xi_1, \dots, \xi_k$ ($k \in \mathbb{N}$) be a number of sections of NS. Define the *fibre derivative* of Π with respect to $(\xi_1, \dots, \xi_k)$ inductively by setting the fibre derivative of Π with respect to $(\emptyset)$ equal to Π and if the fibre derivative with respect to $(\xi_1, \dots, \xi_{(k-1)})$ is the bivector Λ then the fibre derivative of Π with respect to $(\xi_1, \dots, \xi_k)$ is given by

$$\frac{d}{dt}|_{t=0}(\phi_t^{I(\xi_k)})^*(\Lambda) = [I(\xi_k), \Lambda]_{SN}$$

where $\phi_t^{I(\xi_k)}$ is the flow generated by the vector field $I(\xi_k) \in \Gamma(TE)$. The structure maps $(\lambda_n^\sigma)_{n\in\mathbb{N}}$ are only sensitive to the restrictions of all fibre derivatives of Π restricted to S.

The following description of the homotopy Lie algebroid associated to (S, σ) can be found in [**CF**]:

Lemma 3.11. *Let S be a submanifold of a Poisson manifold (M,Π) and σ an embedding of the normal bundle NS of S in M into M such that the restriction to S is the identity. Denote the projection $NS \to S$ by p and the Poisson bivector field on NS inherited from (M,Π) via σ by Π_σ.*

Then the homotopy Lie algebroid of (S,σ) in (M,Π) is determined by the following values of its structure maps $(\lambda_n^\sigma)_{n\in\mathbb{N}}$:

$$\begin{aligned}
\lambda_n^\sigma(\xi_1\otimes\cdots\otimes\xi_n) &= (-1)^n P\left(\xi_1\cdots\xi_n\cdot(\Pi_\sigma)\right),\\
\lambda_n^\sigma(\xi_1\otimes\cdots\otimes\xi_{(n-1)}\otimes f) &= (-1)^n P\left(\xi_1\cdots\xi_{(n-1)}\cdot(\Pi_\sigma^\#(p^*(d_{DR}f)))\right),\\
\lambda_n^\sigma(\xi_1\otimes\cdots\otimes\xi_{(n-2)}\otimes f\otimes g) &= (-1)^n P\left(\xi_1\cdots\xi_{(n-2)}\cdot(\{p^*(f),p^*(g)\}_{\Pi_\sigma})\right).
\end{aligned}$$

Here f and g are functions on S, $\xi_1,\ldots,\xi_n$ are sections of NS and $\xi_1\cdots\xi_k\cdot$ denotes the fibre derivative with respect to $(\xi_1,\ldots,\xi_k)$, see Remark 3.10.

Proof. Part (a) of Lemma 3.7 asserts us that all structure maps λ_n^σ are graded multiderivations of $\Gamma(\wedge NS)$. Since $\Gamma(\wedge NS)$ is locally generated by $\mathcal{C}^\infty(S)$ and $\Gamma(NS)$ it is sufficient to know the values of λ_n^σ on tensor products of elements of $\mathcal{C}^\infty(S)\oplus\Gamma(NS)$. Furthermore λ_n^σ has degree $2-n$ and $\Gamma(\wedge NS)$ is concentrated in non-negative degrees. Consequently λ_n^σ vanishes on all tensor products of elements of $\mathcal{C}^\infty(S)$ and $\Gamma(NS)$ containing strictly more than two factors in $\mathcal{C}^\infty(S)$. Hence only the three cases of tensor products containing two, one or zero functions as listed in the Lemma remain.

That the values are exactly those claimed in the Lemma is a straightforward consequence of the definition of the fiber derivative and λ_n^σ. $\square$

Remark 3.12. Let $E \to S$ be a vector bundle. Consider the family of ideals

$$(\mathcal{V}_{(k)}(E))_{k\geq 1}$$

of $\mathcal{V}(E)$ generated by the powers $(\mathcal{I}_S^k)_{k\geq 1}$ of the vanishing ideal of S in E. Given $k \geq l$ we have natural inclusions

$$\mathcal{I}_S^k \hookrightarrow \mathcal{I}_S^l$$

which induce natural surjectictions of algebras

$$p_{kl} : \mathcal{V}(E)/\mathcal{V}_{(k)}(E) \to \mathcal{V}(E)/\mathcal{V}_{(l)}(E).$$

The pair $((\mathcal{V}(E)/\mathcal{V}_{(k)}(E))_{k\geq 1}, (p_{kl})_{k\geq l})$ forms a projective system of graded algebras. We define the *algebra of formal vector fields* $\mathcal{V}_{\mathrm{for}}(E)$ on E to be the projective limit

$$\mathcal{V}_{\mathrm{for}}(E) := \varprojlim \mathcal{V}(E)/\mathcal{V}_{(k)}(E).$$

This algebra inherits the structure of a Gerstenhaber algebra from $\mathcal{V}(E)$ as follows: Inductively one checks that

$$[\mathcal{V}_{(k)}(E), \mathcal{V}(E)]_{SN} \subset \mathcal{V}_{(k-1)}(E)$$

holds for all $k \geq 1$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two elements in $\mathcal{V}_{\mathrm{for}}(E)$, i.e.

$$\mathcal{X} = (X_1, X_2, \dots) \quad \text{and} \quad \mathcal{Y} = (Y_1, Y_2, \dots)$$

where X_k and Y_k are elements of $\mathcal{V}(E)/\mathcal{V}_{(k)}(E)$ such that

$$p_{kl}(X_k) = X_l \quad \text{and} \quad p_{kl}(Y_k) = X_l$$

holds for all $k \geq l$. Define $[\mathcal{X}, \mathcal{Y}]_{SN}$ to be the element

$$\mathcal{Z} = (Z_1, Z_2, \dots)$$

of $\mathcal{V}_{\mathrm{for}}(E)$ given by

$$Z_k := [\tilde{X}_{(k+1)}, \tilde{Y}_{(k+1)}]_{SN} \bmod \mathcal{V}_{(k)}(NS)$$

where $\tilde{X}_{(k+1)}$ and $\tilde{Y}_{(k+1)}$ are arbitrary representatives of $X_{(k+1)}$ and $Y_{(k+1)}$ in $\mathcal{V}(E)$. It is straightforward to check that this yields a well-defined element $\mathcal{Z}$ of $\mathcal{V}_{\mathrm{for}}(NS)$ and that the corresponding operation $[\cdot,\cdot]_{SN}$ equips $\mathcal{V}_{\mathrm{for}}(NS)$ with the structure of a Gerstenhaber algebra. In particular $\mathcal{V}_{\mathrm{for}}(E)[1]$ is a graded Lie algebra. By construction of $[\cdot,\cdot]_{SN}$ the natural morphism $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ is a morphism of Gerstenhaber algebras.

Recall that the image of a multivector field under the morphism $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ depends only on its values in an arbitrary small neighbourhood of S in E: If two multivector fields X and X' coincide on an open neighbourhood of S in E, their difference is an element of $\mathcal{V}_{(k)}(E)$ for all $k \geq 1$, i.e. all the projections to $\mathcal{V}(E)/\mathcal{V}_{(k)}(E)$ vanish.

Consider the graded algebra $\Gamma(\wedge E)$. In Remark 3.2 an inclusion $I : \Gamma(\wedge E) \to \mathcal{V}(E)$ was constructed. By definition of the projective limit this inclusion yields a morphism $\tilde{I} : \Gamma(\wedge E) \to \mathcal{V}_{\mathrm{for}}(E)$. The fact that $I(\Gamma(\wedge E))$ is an abelian Lie subalgebra of $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$ implies that $\tilde{I}(\Gamma(\wedge E))$ is an abelian Lie subalgebra of $(\mathcal{V}_{\mathrm{for}}(E)[1], [\cdot,\cdot])$. Furthermore there is a morphism $\tilde{P} : \mathcal{V}_{\mathrm{for}}(E) \to \Gamma(\wedge E)$ given componentwise by

$$\wedge \operatorname{pr} \circ p_{k1} : \mathcal{V}(E)/\mathcal{V}_{(k)}(E) \to \mathcal{V}(E)/\mathcal{V}_{(1)} \cong \Gamma(\wedge T|_S E) \to \Gamma(\wedge E).$$

Since $\tilde{P} \circ \tilde{I} = \mathrm{id}$, $\tilde{I}$ is injective, $\tilde{P}$ is surjective and $\tilde{I} \circ \tilde{P}$ is a projection.

LEMMA 3.13. *Given a vector bundle $E \to S$, the graded vector space $\mathcal{V}_{\mathrm{for}}(E)[1]$ splits into an abelian Lie subalgebra $\tilde{I}(\Gamma(\wedge E)[1])$ and a graded Lie subalgebra given by the kernel of*

$$\tilde{P}[1] : \mathcal{V}_{\mathrm{for}}(E)[1] \to \Gamma(\wedge E)[1].$$

Consequently the triple

$$((\mathcal{V}_{\mathrm{for}}(E)[1], [\cdot,\cdot]_{SN}), \tilde{I}(\Gamma(\wedge E)), \tilde{P})$$

is a V-algebra, see Definition 3.1 in Chapter 2.

PROOF. The proof can be copied mutatis mutandis from the proof of Lemma 3.13. □

LEMMA 3.14. *Let $E \to S$ be a vector bundle equipped with a Poisson bivector field Π. Denote the image of Π under $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ by $\tilde{\Pi}$.*

Then the higher derived brackets

$$\begin{array}{rcl} \tilde{\kappa}_n : S^n(\Gamma(\wedge E)[1]) & \to & \Gamma(\wedge E)[2] \\ \xi_1 \otimes \cdots \otimes \xi_n & \mapsto & \tilde{P}\left([[\cdots[[\tilde{\Pi}, \tilde{I}(\xi_1)]_{SN}, \tilde{I}(\xi_2)]_{SN} \cdots], \tilde{I}(\xi_n)]_{SN}\right) \end{array}$$

define the structure of an $L_\infty[1]$-algebra on $\Gamma(\wedge E)[1]$. This is equivalent to the structure of an L_∞-algebra on $\Gamma(\wedge E)$. We denote the associated family of structure maps by

$$\left(\kappa_n : \wedge^n \Gamma(\wedge E) \to \Gamma(\wedge E)[2-n]\right)_{n \in \mathbb{N}}.$$

Furthermore the structure maps $(\kappa_n)_{n\in\mathbb{N}}$ are identical to the structure maps $(\lambda_n)_{n\in\mathbb{N}}$ introduced in Corollary 3.4.

PROOF. The first part of the proof can be copied mutatis mutandis from the proof of Corollary 3.4: Since the map $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ is a morphism of Gerstenhaber algebras, $\tilde{\Pi}$ is a Maurer–Cartan element of the graded Lie algebra $(\mathcal{V}_{\mathrm{for}}(E)[1], [\cdot,\cdot]_{SN})$. By Lemma 3.3 $\Gamma(\wedge NS)[1]$ is part of a V-algebra structure on $(\mathcal{V}_{\mathrm{for}}(E)[1], [\cdot,\cdot]_{SN})$, hence the higher derived brackets equip $\Gamma(\wedge NS)[1]$ with the structure of a $L_\infty[1]$-algebra.

The identity of this $L_\infty[1]$-algebra to the one introduced in Lemma 3.3 follows from the fact that one can reconstruct all fibre derivatives of Π at S from the image of Π under $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$. To be more precise, suppose we want to reconstruct the fibre derivative of Π with respect to the tuple $(\xi_1, \dots, \xi_k)$ of elements in $\Gamma(NS)$ from

$$\tilde{\Pi} = (\Pi + \mathcal{V}_{(1)}(E), \Pi + \mathcal{V}_{(2)}(E), \dots).$$

Choose any representative Λ of $\Pi + \mathcal{V}_{(k+1)}(E)$ in $\mathcal{V}(E)$ and compute its fibre derivative with respect to $(\xi_1, \dots, \xi_k)$:

$$[\xi_k, [\cdots[\xi_2, [\xi_1, \Lambda]_{SN}]_{SN} \cdots]_{SN}]_{SN} = [\xi_k, [\cdots[\xi_1, \Pi]_{SN}]_{SN} \cdots]_{SN}]_{SN} + \mathcal{V}_{(1)}(E).$$

Consequently the fibre derivative of Λ with respect to $(\xi_1, \dots, \xi_k)$ evaluated at S is equal to the fibre derivative of Π with respect to $(\xi_1, \dots, \xi_k)$ evaluated at S. $\square$

THEOREM 3.15. *Let S be a submanifold of a Poisson manifold (M, Π). Suppose σ_0 and σ_1 are two embeddings of the normal bundle of S in M into M such that their restrictions to S are equal to id_S.*

Then the two L_∞-algebra structures $(\lambda_n^{\sigma_0})_{n\in\mathbb{N}}$ and $(\lambda_n^{\sigma_1})_{n\in\mathbb{N}}$ associated to (S, σ_0) and (S, σ_1) respectively are isomorphic.

REMARK 3.16. Theorem 3.15 can be found in [**OP**] for the case of a coisotropic submanifold of a symplectic manifold. The case of an arbitrary submanifold of a Poisson manifold was treated in [**CS**]. We essentially followed the proof given in [**CS**].

PROOF. As mentioned in Remark 2.10 there is an isotopy of embeddings

$$\Sigma : NS \times [0,1] \to M$$

such that the restrictions to $NS \times \{0\}$ and $NS \times \{1\}$ coincides with σ_0 and σ_1 respectively. We denote the restriction of Σ to $NS \times \{t\}$ for $t \in [0,1]$ by σ_t and the images of σ_t by V_t. By Lemma 6 in Chapter 6 there is an open neighbourhood U of S in M such that $U \subset \sigma_t(NS)$ holds for arbitrary $t \in [0,1]$. In particular the image of $NS \times [0,1]$ under

$$\Sigma \times \mathrm{id} : NS \times [0,1] \to M \times [0,1], \quad (n,t) \mapsto (\Sigma(n,t),t)$$

contains $U \times [0,1]$. Since $\Sigma \times \mathrm{id}$ is an embedding, we can define a smooth map

$$\Gamma : U \times [0,1] \to NS$$

which is given by $\sigma_t^{-1}|_U$ for fixed $t \in [0,1]$. Now we apply Lemma 6 again and find an open neighbourhood V of S in NS such that

$$V \subset \sigma_t^{-1}|_U(U)$$

holds for all $t \in [0,1]$. Consider the isotopy of embeddings

$$\Theta : V \times [0,1] \xrightarrow{\Sigma} U \xrightarrow{\sigma_0^{-1}} NS.$$

The restriction of Θ to $V \times \{0\}$ coincides with id_V and the restriction to $S \times \{t\}$ is equal to id_S for arbitrary $t \in [0,1]$. The composition of Θ with σ_0 is the restriction of Σ to $V \times [0,1]$.

The manifold M is equipped with a Poisson bivector field Π. All the embeddings σ_t give rise to a Poisson bivector field on NS defined by $\Pi_t := (\sigma_t)^*(\Pi|_{V_t})$. The identity

$$\sigma_t|_V = (\sigma_0 \circ \Theta_t)|_V$$

implies

$$\Pi_t|_V = (\Theta_t)^* \left(\Pi_0|_{(\sigma_0^{-1} \circ \sigma_t)(V)} \right).$$

Differentiating the smooth one-parameter family of locally defined diffeomorphisms Θ_t yields a smooth one-parameter family of locally defined vector fields $(X_t)_{t \in [0,1]}$. Pulling back X_t along Θ_t defines a smooth one-parameter family of vector fields

$$(Y_t := (\Theta_t)^*(X_t))_{t \in [0,1]}$$

defined on V. The flow equation

$$\frac{d}{dt}|_{t=s}\Theta_t = X_s|_{\Theta_s}, \quad \Theta_0 = \mathrm{id}_V$$

is equivalent to

$$\frac{d}{dt}|_{t=s}(\Theta_t)^*(\cdot) = [Y_s, (\Theta_s)^*(\cdot)]_{SN}, \quad (\Theta_0)^* = \mathrm{id}\,.$$

The pull backs by the locally defined diffeomorphisms Θ_t map locally defined smooth functions in the vanishing ideal of S in E to themselves. This implies that $((\Theta_t)^*)_{t\in[0,1]}$ induces an automorphism Ψ_t of $\mathcal{V}_{\mathrm{for}}(E)$. Consider an element $\mathcal{X} \in \mathcal{V}_{\mathrm{for}}(E)$ given by

$$(X_1, X_2, \dots).$$

We define $\mathcal{Y} =: \Psi_t(\mathcal{X})$ with components

$$(Y_1, Y_2, \dots)$$

as follows: pick a representative $\tilde{X}_k$ for the class X_k in $\mathcal{V}(E)/\mathcal{V}_{(k)}(E)$. Apply $(\Theta_t)^*$ to X_k where possible. The result is defined on an open neighbourhood K of S in E. Choose an even smaller open neighbourhood L of S in E and extend $(\Theta_t)^*(X_k)$ to be zero outside of K while leaving it unchanged on L. Denote the resulting multivector field by $\tilde{Y}_k$ and set $Y_k := \tilde{Y}_k + \mathcal{V}_{(k)}(E)$.

To see that the element $\mathcal{Y}$ is well-defined recall that the image of a multivector field under $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ only depends on its values on an arbitrary small open neighbourhood of S in E and that $(\Theta_t)^*$ maps locally defined elements of $\mathcal{V}_{(k)}(E)$ to locally defined elements of $\mathcal{V}_{(k)}(E)$.

Denote the image of $(Y_t := ((\Theta_t^*)X_t)_{t\in[0,1]}$ under $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ by $(\mathcal{Y}_t)_{t\in[0,1]}$. The flow equation for $(\Theta_t^*)_{t\in[0,1]}$ implies that the identity

$$\frac{d}{dt}|_{t=s}(\Psi_t(\cdot)) = [\mathcal{Y}_s, \Psi_s(\cdot)]_{SN}, \quad \Psi_0 = \mathrm{id}$$

holds for all elements of $\mathcal{V}_{\mathrm{for}}(E)$ and arbitrary $s \in [0,1]$. Denote the images of Π_t under $\mathcal{V}(E) \to \mathcal{V}_{\mathrm{for}}(E)$ by $\tilde{\Pi}_t$. The identity

$$\Pi_t|_V = (\Theta_t)^* \left(\Pi_0|_{(\sigma_0^{-1}\circ\sigma_t)(V)} \right).$$

implies

$$\tilde{\Pi}_t = \Psi_t(\tilde{\Pi}_0)$$

for arbitrary $t \in [0,1]$.

To summarize the situation we have a one-parameter family of Maurer–Cartan elements $(\tilde{\Pi}_t)_{t\in[0,1]}$ of the V-algebra

$$(\mathcal{V}_{\mathrm{for}}(NS)[1], [\cdot,\cdot]_{SN}, \Gamma(\wedge NS)[1], \tilde{P})$$

and a one-parameter family of automorphisms $(\Psi_t)_{t\in[0,1]}$ generated by a one-parameter family of inner derivations $([\mathcal{Y}_t, \cdot]_{SN})_{t\in[0,1]}$ such that the relation

$$\tilde{\Pi}_t = \Psi_t(\tilde{\Pi}_0)$$

is satisfied for all $t \in [0,1]$.

In order to apply Theorem 3.7, Chapter 2 we have to verify the condition

$$\tilde{P}(\mathcal{Y}_t) = 0$$

and that solutions to the equation

$$\frac{d}{dt}|_{t=s}\xi_t = \tilde{P}([\mathcal{Y}_s, \xi_s]_{SN}), \quad \xi_0 \in \Gamma(\wedge NS)$$

on the space of sections of the pull back of $NS \to S$ along $S \times [0,1] \to S$ are unique.

To verify the first condition observe that the smooth one-parameter family of diffeomorphisms $(\Theta_t)_{t\in[0,1]}$ leaves S invariant, hence the restriction of the smooth one-parameter family of vector fields $(X_t)_{t\in[0,1]}$ to S vanishes. Consequently X_t lies in $\mathcal{V}_{(1)}(E)$ for arbitrary $t \in [0,1]$. Since Θ_t preserves the vanishing ideal $\mathcal{I}_S$ the vector field $Y_t := \Theta_t^*(X_t)$ also lies in $\mathcal{V}_{(1)}(E)$ for arbitrary $t \in [0,1]$. This implies that $\tilde{P}(\mathcal{Y}_t) = 0$ holds for all $t \in [0,1]$.

That

$$\frac{d}{dt}|_{t=s}\xi_t = \tilde{P}([\mathcal{Y}_s, \xi_s]_{SN}), \quad \xi_0 \in \Gamma(\wedge NS)$$

has a unique solution and admits a family of integrating automorphisms is seen as follows: first check that the one-parameter family of derivations $D_s := \tilde{P}([\mathcal{Y}_s, \cdot]_{SN})$ annihilates $\mathcal{C}^\infty(S)$ because $\mathcal{Y}_s$ lies in the kernel of $\tilde{P}$. Hence $(D_s)_{s\in[0,1]}$ is $\mathcal{C}^\infty(S)$-linear and this implies that it acts fibrewise for all $s \in [0,1]$. Since D_s is a graded derivation with respect to the wedge product it is enough to know its restriction to $\Gamma(NS)$. By degree reasons $D_s(\Gamma(NS)) \subset \Gamma(NS)$ and because D_s acts fibrewise its restriction to $\Gamma(NS)$ is given by fibrewise linear derivations of N_xS for all $x \in S$, i.e. we obtain a smooth one-parameter family of sections of $\mathrm{End}((NS)$. Any such smooth one-parameter family integrates to an smooth one-parameter family of sections of $GL_+(NS)$. The natural extension of this family to a family of automorphisms of $\wedge NS$ yields the one-parameter family of automorphisms of $\Gamma(\wedge NS)$ that uniquely integrates the Cauchy problem from above. $\square$

CHAPTER 4

The BFV-Complex

The aim of this Chapter is to explain the construction of a differential graded Poisson algebra associated to coisotropic submanifolds which is known as the BFV-complex. This structure was originally introduced by Batalin, Fradkin and Vilkovsiky in order to quantize field theories with complicated symmetries ([**BF**], [**BV**]). Later on it was given an interpretation in terms of homological algebra by Stasheff, see [**Sta2**]. Bordemann and Herbig ([**B**], [**He**]) adapted the construction to arbitrary coisotropic submanifolds of finite-dimensional Poisson manifolds.

Section 1 explains a lifting procedure of Poisson structures to a Poisson bivector on a certain smooth graded manifold. In particular this construction yields a conceptual understanding of the BFV-bracket which was originally presented in [**Sch1**]. In Section 2 the BFV-complex is introduced and its main properties are established: we show that its cohomology is isomorphic to the Lie algebroid cohomology (Lemma 2.19), and establish the invariance of a germ version of the BFV-complex – see Theorem 2.31. The dependence of the BFV-complex on the choices involved in its construction was clarified in [**Sch2**]. The third Section connects the BFV-complex to the homotopy Lie algebroid which was introduced in Section 3 in Chapter 3. More precisely, Theorem 3.6 asserts that these two structures are L_∞ quasi-isomorphic.

1. Lifting

Remark 1.1. Given a finite rank vector bundle $\mathcal{F} \to F$ over a smooth finite-dimensional manifold F, the vector space $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ is equipped with a lot of structures: First it is a bigraded algebra with respect to the bigrading

$$\Gamma^{(p,q)}(\wedge(\mathcal{F}\oplus\mathcal{F}^*)) := \Gamma((\wedge^p\mathcal{F})\otimes(\wedge^q\mathcal{F}^*)).$$

We refer to p/q as the *ghost degree/ghost-momentum degree*. The *total degree* is given by the difference between the ghost degree and the ghost-momentum degree. We denote the component of total degree k by $\Gamma^k(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$. The total degree equips $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ with the structure of a graded algebra. Moreover let $\Gamma_{\geq r}(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ be the ideal $\Gamma((\wedge\mathcal{F})\otimes(\wedge^{\geq r}\mathcal{F}^*))$.

The contraction between $\mathcal{F}$ and $\mathcal{F}^*$ induces a symmetric pairing on $\Gamma(\mathcal{F}\oplus\mathcal{F}^*)$. Extending this pairing to a graded skew-symmetric (with respect to the total degree) graded biderivation of degree 0 on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ yields a graded Poisson bracket $[\cdot,\cdot]_G$ on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$.

The graded algebra $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ can be interpreted as the algebra of smooth functions on $\mathcal{F}^*[1]\oplus\mathcal{F}[-1]$.

REMARK 1.2. Following the general theory outlined in Section 4, Chapter 2, the space of multiderivations of the graded algebra $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ is given by

$$\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))=\mathcal{S}_{\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))}(\mathrm{Der}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[-1])$$

i.e. the graded symmetric algebra generated by the algebra of graded derivations $\mathrm{Der}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[-1]$ of the graded algebra $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ as a graded module over $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$. By 4.14 in Chapter 2 $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1]$ carries a graded Lie bracket $[\cdot,\cdot]_{SN}$.

The bidegree on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ given by the ghost and ghost-momentum degree respectively induces a bidegree on $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$: the tensor product of a number of copies of $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ is equipped with a bidegree given by the sum of the bidegrees of the individual factors. A multiderivation yields a linear map from such tensor products to elements of $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$. The bidegree of the multiderivation is defined to be the bidegree of this map. Observe that $[\cdot,\cdot]_{SN}$ is additive with respect to this bidegree. We denote the ideal of $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$ generated by elements of bidegree greater or equal to (k,l) by $\mathcal{D}^{(k,l)}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$.

An alternative point of view on $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$ is to realize it as the algebra of smooth functions on the smooth graded manifold $T^*[1](\mathcal{F}^*[1]\oplus\mathcal{F}[-1])$, see Lemma 4.10 in Chapter 2.

Lemma 4.18 in Chapter 2 asserts that the quadruple

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1],[\cdot,\cdot]_{SN},\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))[1],\mathrm{pr}[1])$$

is a V-algebra. A Poisson multivector field Z on $\mathcal{F}^*[1]\oplus\mathcal{F}[-1]$ is by definition a Maurer–Cartan element of the graded Lie algebra $(\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1],[\cdot,\cdot]_{SN})$. Any such element Z yields the structure of a P_∞-algebra on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$, see Lemma 4.18, Chapter 2.

On the other hand, any P_∞-algebra gives rise to a graded multiderivation of $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ and the vanishing of the associated family of Jacobiators is equivalent to the Maurer–Cartan equation for this graded multiderivation. In particular the graded Poisson bracket $[\cdot,\cdot]_G$ on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)$ corresponds to a unique Poisson bivector field on $\mathcal{F}^*[1]\oplus\mathcal{F}[-1]$ which we denote by G.

LEMMA 1.3. *Given a vector bundle $\mathcal{F}\to F$ the cohomology of the complex*

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\wedge\mathcal{F}^*)))[1],[G,\cdot]_{SN})$$

is isomorphic to the graded algebra $\mathcal{V}(F)[1]$).

PROOF. As described in Remark 1.2 we can interpret $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\wedge\mathcal{F}^*)))$ as the algebra of smooth functions on the smooth graded manifold $T^*[1](\mathcal{F}^*[1]\oplus\mathcal{F}[-1])$. In particular every connection ∇ on $\mathcal{F}\to F$ yields an induced connection ∇ on

$$\wedge(\mathcal{F}\oplus\mathcal{F}^*)\to F$$

and consequently an isomorphism

$$\Psi_\nabla : \mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \wedge\mathcal{F}^*))) \cong \Gamma(\wedge TF \otimes \wedge(\mathcal{F} \oplus \mathcal{F}^*) \otimes \mathcal{S}(\mathcal{F}^* \oplus \mathcal{F})) =: \mathcal{A}$$

where the additional components $\mathcal{F}^*$ and $\mathcal{F}$ are declared to be of degree 0 and 2 respectively. To make the distinction to the original $\mathcal{F}^*$ and $\mathcal{F}$ clearer we denote these two additional copies by $\mathcal{F}^*[0]$ and $\mathcal{F}[-2]$ from now on.

Equip $\mathcal{A}[1]$ with the unique structure of a differential graded Lie algebra

$$(\mathcal{A}[1], \tilde{Q} := [\tilde{G}, \cdot]_\nabla, [\cdot, \cdot]_\nabla)$$

such that $\Psi_\nabla[1]$ becomes an isomorphism of differential graded Lie algebras, where $\tilde{G} := \Psi_\nabla(G)$.

Because $[\tilde{G}, \cdot]_\nabla$ is a graded derivation of $\mathcal{A}$ of degree $+1$ it suffices to know it on generators of $\mathcal{A}$. These are given by functions and vector fields on F, and sections of the two copies of $\mathcal{F}$ and $\mathcal{F}^*$ respectively. Observe that the $\tilde{G}$ corresponds to the identity-section under the identifitication $\mathcal{F}^*[0] \otimes \mathcal{F}[-2] \cong \mathrm{End}(\mathcal{F})[-2]$. The identities

$$\tilde{Q}(\Gamma(\mathcal{F}[-2])) = 0 = \tilde{Q}(\Gamma(\mathcal{F}^*[0]))$$

are clear: all multivector fields involved act in the vertical direction and are constant along the fibres of $\mathcal{F} \oplus \mathcal{F}^*$. Hence the Schouten-Nijenhuis bracket vanishes and this implies that $\Gamma(\mathcal{F}[-2])$ and $\Gamma(\mathcal{F}^*[0])$ get annihilated by $\tilde{Q}$. A computation in local charts shows that

$$\tilde{Q}(\mathcal{C}^\infty(S)) = 0 = \tilde{Q}(\Gamma(TS))$$

holds. The second equality is true because ∇ is metric with respect to the pairing between $\mathcal{F}$ and $\mathcal{F}^*$. The action of $\tilde{Q}$ on $\Gamma(\mathcal{F})$ and $\Gamma(\mathcal{F})$ is given by

$$\begin{aligned} \Gamma(\mathcal{F}) &\xrightarrow{\mathrm{id}} \Gamma(\mathcal{F}[-2]) \quad \text{and} \\ \Gamma(\mathcal{F}^*) &\xrightarrow{\mathrm{id}} \Gamma(\mathcal{F}^*[0]) \quad \text{respectively.} \end{aligned}$$

Remarkably the differential $\tilde{Q}$ on $\mathcal{A}$ does not depend on the connection ∇.

There is an inclusion i_∇ of $\mathcal{V}(F)$ in $\mathcal{A}$ and a natural projection p from $\mathcal{A}$ to $\mathcal{V}(F)$ satisfying $p \circ i_\nabla = \mathrm{id}$. Observe that i_∇ corresponds to the horizontal lift with respect to ∇ while p corresponds to the natural projection

$$\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*))) \to \mathcal{V}(F).$$

From the description of $\tilde{Q}$ given above it is clear that i_∇ and p are morphisms between the complexes $(\mathcal{A}[1], [\tilde{G}, \cdot]_\nabla)$ and $(\mathcal{V}(F)[1], 0)$ respectively.

Next we introduce a differential $\tilde{H}$ on $\mathcal{A}$. We declare the action of $\tilde{H}$ on $\Gamma(\mathcal{F}[-2])$ and $\Gamma(\mathcal{F}[0])$ by

$$\begin{aligned} \Gamma(\mathcal{F}[-1]) &\xrightarrow{\mathrm{id}} \Gamma(\mathcal{F}) \quad \text{and} \\ \Gamma(\mathcal{F}^*[0]) &\xrightarrow{\mathrm{id}} \Gamma(\mathcal{F}^*) \quad \text{respectively,} \end{aligned}$$

set it zero on $\mathcal{C}^\infty(F)$, $\Gamma(TF)$, $\Gamma(\mathcal{F})$ and $\Gamma(\mathcal{F}^*)$ and extend it to all of $\mathcal{A}$ as a graded derivation of degree -1. The property $\tilde{H} \circ \tilde{H} = 0$ is clear from the definition, as are the identities $\tilde{H} \circ i_\nabla = 0$ and $p \circ \tilde{H} = 0$.

Moreover

$$[\tilde{H}, \tilde{Q}] = \tilde{H} \circ \tilde{Q} + \tilde{Q} \circ \tilde{H}$$

is a graded derivation of degree 0 that is the identity on $\Gamma(\mathcal{F})$, $\Gamma(\mathcal{F}^*)$, $\Gamma(\mathcal{F}[-2])$ and $\Gamma(\mathcal{F}^*[0])$ and zero on $\mathcal{C}^\infty(F)$ and $\Gamma(TF)$. This implies that $[\tilde{H}, \tilde{Q}]$ is given by the multiplication of elements of $\mathcal{A}[1]$ that are homogeneous in $\wedge\mathcal{F}$, $\wedge\mathcal{F}^*$, $\mathcal{S}(\mathcal{F}^*[0])$ and $\mathcal{S}(\mathcal{F}[-2])$ by the sum of their polynomial degrees along all these fibre directions. Normalizing $\tilde{H}$ by this factor on homogeneous elements yields a coboundary operator H that satisfies

$$[H, \tilde{Q}] = \mathrm{id} - i_\nabla \circ p,$$

i.e. H is a homotopy between id and $i_\nabla \circ p$. We denote the corresponding homotopy for $[G, \cdot]_{SN}$ by H_∇.

Consequently i_∇ and p induce inverse algebra isomorphisms between $H(\mathcal{A}, \tilde{Q})$ and $\mathcal{V}(F)$. Since the complex

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \wedge\mathcal{F}^*)))[1], [G, \cdot]_{SN})$$

is isomorphic to $(\mathcal{A}, \tilde{Q})$ this finishes the proof. □

Proposition 1.4. *Given a vector bundle $\mathcal{F} \to F$, every choice of connection ∇ gives rise to an L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ from the graded Lie algebra*

$$(\mathcal{V}(F)[1], [\cdot, \cdot]_{SN})$$

to the differential graded Lie algebra

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$$

which extends the horizontal lift with respect to ∇.

Proof. Recall the proof of Lemma 1.3: With the help of a connection on $\mathcal{F} \to F$ we equipped

$$\mathcal{A}[1] := \Gamma(\wedge(TF) \otimes \wedge(\mathcal{F} \oplus \mathcal{F}^*) \otimes \mathcal{S}(\mathcal{F}^* \oplus \mathcal{F}))[1]$$

with the structure of a differential graded Lie algebra $(\mathcal{A}[1], \tilde{Q}, [\cdot, \cdot]_\nabla)$ isomorphic to

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN}).$$

Furthermore we obtained contraction data – see Definition 2.1 in Chapter 2 –

$$(\mathcal{V}(F), 0) \underset{p}{\overset{i_\nabla}{\rightleftarrows}} (\mathcal{A}, \tilde{Q}), H.$$

According to Theorem 2.2 in Chapter 2 these data can be used to perform homological transfer of L_∞-algebra structures along p. Starting with the differential graded

Lie algebra $(\mathcal{A}[1], \tilde{Q}, [\cdot,\cdot]_\nabla)$ one constructs an L_∞ quasi-isomorphic L_∞-algebra on $\mathcal{V}(F)[1]$ (with zero differential) together with an L_∞ quasi-isomorphism

$$\mathcal{K}_\nabla : \mathcal{V}(F)[1] \rightsquigarrow \mathcal{A}[1].$$

The binary operation of the induced L_∞-algebra on $\mathcal{V}(F)[1]$ is given by

$$p\left([i_\nabla(\cdot), i_\nabla(\cdot)]_\nabla\right).$$

The difference between $[i_\nabla(\cdot), i_\nabla(\cdot)]_{SN}$ and $i_\nabla([\cdot,\cdot]_{SN})$ is given by the curvature $R_\nabla(\cdot,\cdot)$ interpreted as a fibrewise acting vector field. As such its restriction to F vanishes. Consequently the induced binary operation on $\mathcal{V}(E)[1]$ is – up to a sign shift coming form the décalage-isomorphism – equal to the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$.

In order to prove that all higher induced structure maps vanish, we introduce a bigrading on $\mathcal{A}$: it is given by the difference between the polynomial degrees along $\mathcal{F}$ and along $\mathcal{F}^*[0]$ on the one hand and by the difference between the polynomial degrees along $\mathcal{F}^*$ and along $\mathcal{F}[-2]$ on the other hand. The corresponding bidegree on $\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$ is the bidegree introduced in Remark 1.2. It is bounded from above and the projection p annihilates all elements not of bidegree $(0,0)$. Let $\mathcal{A}^{(k,l)}(E)$ be the ideal in $\mathcal{A}(E)$ generated by all elements of bidegree greater or equal to (k,l).

According to Remark 2.7, Chapter 2 the structure maps of the induced L_∞-algebra on $\mathcal{V}(E)[1]$ are all given diagrammatically by decorated oriented trivalent trees whose interior vertices are decorated by $[\cdot,\cdot]_\nabla$, whose interior edges (i.e. edges not connected to any leaf or the root) are decorated by $-H$, whose leaves (i.e. exterior vertices with edges pointing away from them) are decorated by i_∇ and whose root is decorated by p. Denote the number of interior vertices by e. We claim that if we replace p at the root by id the image of $\mathcal{V}(E)[1]^{\otimes e}$ under the map associated to the tree lies in $\mathcal{A}^{(e-1,e-1)}$ and consequently all contributions form trees with more than one interior vertex are annihilated by p.

Observe that

- the image of i_∇ lies in $\mathcal{A}^{(0,0)}$,
- $[\cdot,\cdot]_\nabla$ is of bidegree $(0,0)$ and
- H increases the bidegree by $(1,1)$.

This implies that the image of a tree with e interior vertices lies in bidegree $(e-1,e-1)$.

By Proposition 2.10 in Chapter 2 we obtain an L_∞-morphism between $\mathcal{V}(F)[1]$ equipped with the induced L_∞-algebra structure and $(\mathcal{A}[1], \tilde{Q}, [\cdot,\cdot]_{SN})$ which is isomorphic to

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN}).$$

Moreover we verified that the induced structure is simply $(\mathcal{V}(F)[1], [\cdot,\cdot]_{SN})$ which implies that we obtain an L_∞ morphism

$$\mathcal{L}_\nabla : (\mathcal{V}(F)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$$

extending the horizontal lift i_∇. Since the horizontal lift i_∇ is a chain map from $(\mathcal{V}(F), 0)$ to $(\mathcal{A}, \tilde{Q})$ that induces an isomorphism on cohomology, the L_∞-morphism is in fact an L_∞ quasi-isomorphism. □

REMARK 1.5. Although the induced L_∞-algebra structure on $\mathcal{V}(F)[1]$ is the usual one, the L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ is a non-trivial perturbation of the horizontal lift i_∇. By Proposition 2.10, Chapter 2 this L_∞ quasi-isomophism is given in terms of oriented trivalent trees whose trivalent vertices are decorated by $[\cdot,\cdot]_{SN}$, whose interior edges (i.e. those edges not connected to any leaf or the root) and the edge pointing to the root are decorated by $-$ the homotopy and whose leaves (i.e. exterior vertices with edges pointing away from them) are decorated by the horizontal lift i_∇. All these maps are graded multiderivations, therefore it suffices to know $\mathcal{L}_\nabla$ on tensor products of functions and vector fields on F. Since $[\cdot,\cdot]_{SN}$ and the homotopy decrease the multivector field degree by 1 the component

$$(\mathcal{L}_\nabla)_n : \wedge^n(\mathcal{V}(F)[1]) \to \mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[2-n]$$

of $\mathcal{L}_\nabla$ maps a tensor product of multivector fields of total multivector field degree N to a multiderivation of multiderivation degree $N-2(n-1)$. The multiderivation degree is concentrated in non-negative degrees, hence all tensor products of total multivector field degree $N < 2(n-1)$ get mapped to zero. Because we only consider tensor products of functions and vector fields we know that $n \geq N$. In summary

$$n \geq N \geq 2(n-1)$$

must hold which only admits solutions for $n \leq 2$. We know that $(\mathcal{L}_\nabla)_1 = i_\nabla$. For $n = 2$ the only possible value for N is two, i.e. we have to calculate

$$(\mathcal{L}_\nabla)_2(X \otimes Y) = -H_\nabla([i_\nabla(X), i_\nabla]_{SN})$$

for arbitrary vector fields X and Y on F. Compute

$$\begin{aligned} -H_\nabla([i_\nabla(X), i_\nabla(Y)]_{SN}) &= -H_\nabla(R_\nabla(X,Y) + i_\nabla([X,Y]_{SN}) \\ &= -H_\nabla(R_\nabla(X,Y)) \\ &= -\mathcal{R}_\nabla(X,Y). \end{aligned}$$

Here $R_\nabla(\cdot,\cdot)$ is the curvature of the induced connection on $\wedge(\mathcal{F}\oplus\mathcal{F}^*)$ seen as a differential form with values in fibrewise acting vector fields and $\mathcal{R}_\nabla(\cdot,\cdot)$ denotes the curvature of the connection ∇ on $\mathcal{F} \to F$ interpreted as an element of $\Omega^2(F, \mathrm{End}(F)) = \Omega^2(F, \mathcal{F}^* \otimes \mathcal{F})$.

The image of $Z_1 \otimes \cdots \otimes Z_n$ is given by

- lifting Z_1 and Z_n to vector fields on $\wedge(\mathcal{F}\oplus\mathcal{F}^*)$,

- pulling back the curvature of ∇ to a two-form $\mathcal{R}_\nabla(\cdot,\cdot)$ on $\wedge(\mathcal{F}\oplus\mathcal{F}^*)$ with values in $\mathcal{F}^*\otimes\mathcal{F}$ and
- contracting pairs of the lifting multivector fields with copies of the two-form $\mathcal{R}_\nabla(\cdot,\cdot)$.

COROLLARY 1.6. *Let $\mathcal{F}\to F$ be a vector bundle over a Poisson manifold (F,Π) and denote the natural projection*

$$\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))\to\mathcal{C}^\infty(F)$$

by π.

Then there is a graded biderivation $[\cdot,\cdot]_{BFV}$ on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ of degree 0 with the following properties:

(a) *The biderivation $[\cdot,\cdot]_{BFV}$ satisfies the graded Jacobi identity, i.e. it is a graded Poisson bracket on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ of degree 0.*
(b) *The restriction of $\pi\circ[\cdot,\cdot]_{BFV}$ to $\mathcal{C}^\infty(F)\times\mathcal{C}^\infty(F)$ coincides with the Poisson bracket $\{\cdot,\cdot\}_\Pi$ associated to Π.*
(c) *The restriction of $\pi\circ[\cdot,\cdot]_{BFV}$ to $\Gamma(\mathcal{F})\times\Gamma(\mathcal{F}^*)$ coincides with the pairing between $\Gamma(\mathcal{F})$ and $\Gamma(\mathcal{F}^*)$ induced by the natural fibre pairing between $\mathcal{F}$ and $\mathcal{F}^*$.*

PROOF. Choose a connection ∇ on $\mathcal{F}\to F$. By Proposition 1.4 this gives rise to an L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ from the graded Lie algebra $(\mathcal{V}(F)[1],[\cdot,\cdot]_{SN})$ to the differential graded Lie algebra

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1],[G,\cdot]_{SN},[\cdot,\cdot]_{SN}).$$

The Poisson bivector field Π is a Maurer–Cartan element of $(\mathcal{F}[1],[\cdot,\cdot]_{SN})$ and, if

$$\sum_{k\geq 1}\frac{1}{k!}(\mathcal{L}_\nabla)_k(\Pi\otimes\cdots\otimes\Pi)$$

converges, it yields a Maurer–Cartan element Λ of

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1],[G,\cdot]_{SN},[\cdot,\cdot]_{SN}).$$

This is equivalent to the claim that $G+\Lambda$ is a Maurer–Cartan element of

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))[1],[\cdot,\cdot]_{SN}),$$

which corresponds to a P_∞-algebra structure on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$, see Remark 1.2 and Lemma 4.18 in Chapter 2.

We claim that

$$\Lambda:=\sum_{k\geq 1}\frac{1}{k!}(\mathcal{L}_\nabla)_k(\Pi\otimes\cdots\otimes\Pi)$$

converges and that it yields a biderivation with the desired properties.

Convergence follows from the fact that the image of $(\mathcal{L}_\nabla)_k$ lies in $\mathcal{D}^{(k,k)}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$. This was established in the proof of Proposition 1.4. Since the bidegree on

$\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$ is bounded from above, $(\mathcal{L}_\nabla)_k$ vanishes for k big enough and the series above is a finite sum.

The operation $[\cdot,\cdot]_{SN}$ maps bivector fields to trivector fields, while the homotopy on $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$ constructed in the proof of Proposition 1.4 maps trivector fields to bivector fields. Since the construction of the L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ only uses diagrams decorated by the same number of $[\cdot,\cdot]_{SN}$ and homotopies, it maps bivector fields to bivector fields. Hence Λ is a biderivation.

Properties (b) and (c) follow from the observation that

$$G+\Lambda = G + i_\nabla(\Pi) + \mathcal{D}^{(1,1)}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$$

holds. □

Remark 1.7. Our main application of Corollary 1.6 will be the construction of the BFV-bracket in the next Section. We refer to any graded Poisson bracket on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ of degree 0 over a Poisson manifold (F,Π) satisfying properties (a), (b) and (c) as a *BFV-bracket* on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$.

Corollary 1.6 was presented in [**R**] under the assumption that the Poisson bivector field Π on F is symplectic. It was extended to the general case by Herbig later, see [**He**]. However the proof presented here is quite different: the arguments in [**R**] and [**He**] rely on explicit formulae for the lift Λ instead of the quasi-isomorphism $\mathcal{L}_\nabla$ from Proposition 1.4. This conceptual approach was introduced in [**Sch1**]. It has the advantage of clarifying the dependence of the lift on the connection ∇, see [**Sch2**].

Proposition 1.8. *Given a vector bundle $\mathcal{F}\to F$ over a Poisson manifold (M,Π) equipped with two BFV-brackets $[\cdot,\cdot]_{BFV}$ and $[\cdot,\cdot]'_{BFV}$, there is an automorphism of the unital graded algebra $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ which is an isomorphism of graded Poisson algebras*

$$(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)),[\cdot,\cdot]_{BFV}) \xrightarrow{\cong} (\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)),[\cdot,\cdot]'_{BFV}).$$

Remark 1.9. Proposition 1.8 was presented in [**Sch1**] although in a slightly different form.

Proof. Denote the bivector fields on $\mathcal{F}^*[1]\oplus\mathcal{F}[-1]$ corresponding to $[\cdot,\cdot]_{BFV}$ and $[\cdot,\cdot]'_{BFV}$ by $G+\Lambda$ and $G+\Lambda'$ respectively. Observe that the requirement to be a graded biderivation of degree 0 forces every BFV-bracket to be the sum of elements of bidegree (k,k) in $\mathcal{D}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$.

Consider the difference $\Theta=\Lambda-\Lambda'$. The higher derived brackets with respect to Θ yield a biderivation on $\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$ that induces the trivial biderivation on $\mathcal{C}^\infty(F)$ since both $[\cdot,\cdot]_{BFV}$ and $[\cdot,\cdot]_\nabla$ satisfy property (b) from Corollary 1.6. Let $\lambda(0)$,$\lambda'(0)$ and $\theta(0)$ be the components of Λ, Λ' and Θ of bidegree $(0,0)$. Because of

$$0=[G+\Lambda,G+\Lambda]_{SN}+\mathcal{D}^{(0,0)}(\wedge(\mathcal{F}\oplus\mathcal{F}^*)) = 2[G,\lambda(0)]_{SN}+\mathcal{D}^{(0,0)}(\wedge(\mathcal{F}\oplus\mathcal{F}^*))$$

the difference $\theta(0) = \lambda(0) - \lambda'(0)$ satisfies $[G, \theta(0)]_{SN} = 0$. The cohomology class of $\theta(0)$ in

$$H(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*))), [G, \cdot]_{SN})$$

vanishes since the projection of $\theta(0)$ to $\mathcal{V}(F)$ vanishes. Fix a connection ∇ on $\mathcal{F} \to F$. The element $\gamma(0) := H_\nabla(\theta(0))$ satisfies

$$[G, \gamma(0)]_{SN} = \theta(0)$$

and it is a graded derivation of bidegree $(1,1)$. Consequently it acts as a nilpotent derivation on $\mathcal{D}(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$ and integrates to an automorphism $\Gamma(0)$ of the graded Lie algebra

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))[1], [\cdot, \cdot]_{SN}).$$

Set $G + \Lambda^{(1)} := \Gamma(0)(G + \Lambda)$ which is a graded bivector field on $\mathcal{F}^*[1] \oplus \mathcal{F}[-1]$ that satisfies the Maurer–Cartan equation. Moreover

$$\begin{aligned}
\Gamma(0)(G + \Lambda) &= (G + \Lambda) + [\gamma(0), (G + \Lambda)]_{SN} + \mathcal{D}^{(1,1)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*))) \\
&= G + \lambda(0) + [\gamma(0), G]_{SN} + \mathcal{D}^{(1,1)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*))) \\
&= G + \lambda(0) - \theta(0) + \mathcal{D}^{(1,1)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*))) \\
&= G + \lambda'(0) + \mathcal{D}^{(1,1)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))
\end{aligned}$$

and consequently $G + \Lambda^{(1)}$ and $G + \Lambda'$ coincide up to $\mathcal{D}^{(1,1)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$.

Now suppose $G + \tilde{\Lambda}$ and $G + \Lambda'$ coincide up to $\mathcal{D}^{(k,k)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$ with $k > 0$, i.e. the difference $\Theta(k) := \tilde{\Lambda} - \Lambda'$ is an element of $\mathcal{D}^{(k,k)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$. Denote the components of $\Theta(k)$, $\tilde{\Lambda}$ and Λ' of bidegree (k,k) by $\theta(k)$, $\tilde{\lambda}(k)$ and $\lambda'(k)$ respectively. The equation

$$[G + \tilde{\Lambda}, G + \tilde{\Lambda}]_{SN} = 0$$

implies that $[G, \tilde{\lambda}(k)]_{SN}$ can be expressed as a function $F(\tilde{\lambda}(0), \cdots, \tilde{\lambda}(k-1))$ depending on the components of $\tilde{\Lambda}$ in lower bidegrees. Consequently

$$\begin{aligned}
[G, \tilde{\lambda}(k)]_{SN} &= F(\tilde{\lambda}(0), \cdots, \tilde{\lambda}(k-1)) \\
&= F(\lambda'(0), \cdots, \lambda'(k-1)) \\
&= [G, \lambda'(k)]_{SN}
\end{aligned}$$

and hence $[G, \theta(k)]_{SN} = [G, \tilde{\lambda}(k) - \lambda'(k)]_{SN} = 0$. But all elements of bidegree not equal to $(0,0)$ get mapped to zero by the projection $\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$, so the cohomology class of $\theta(k)$ vanishes. Define $\gamma(k) := H_\nabla(\theta(k))$ which acts as a nilpotent derivation. It integrates to an automorphism $\Gamma(k)$ of the graded Lie algebra

$$(\mathcal{D}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))[1], [\cdot, \cdot]_{SN}).$$

Set $G + \Lambda^{(k+1)} := \Gamma(k)(G + \tilde{\Lambda})$ which is a graded bivector field on $\mathcal{F}^*[1] \oplus \mathcal{F}[-1]$ that satisfies the Maurer–Cartan equation. One checks that $G + \Lambda^{(k+1)}$ and $G + \Lambda'$ coincide up to $\mathcal{D}^{(k+1,k+1)}(\Gamma(\wedge(\mathcal{F} \oplus \mathcal{F}^*)))$.

Because the filtration $\mathcal{D}^{(p,q)}(\Gamma(\wedge(\mathcal{F}\oplus\mathcal{F}^*)))$ is bounded from above the procedure stops after finitely many steps and we obtain a family of automorphisms $\Gamma(0),\dots,\Gamma(N)$ satisfying

$$G+\Lambda' = (\Gamma(N)\circ\cdots\circ\Gamma(0))(G+\Lambda).$$

$\square$

2. The BFV-Complex

REMARK 2.1. Similar to the homotopy Lie algebroid (Section 3), the BFV-complex is a structure associated to a coisotropic submanifold S of a Poisson manifold (M,Π) *and* an embedding σ of the normal bundle of S in M into M which restricts to the identity on S.

For the moment we fix such an embedding and work on the normal bundle NS equipped with the Poisson bivector field Π_σ given by the identification $\sigma : NS \xrightarrow{\cong} \sigma(NS)\subset M$ and the restriction of Π to the open submanifold $\sigma(NS)$ of M.

DEFINITION 2.2. Given a vector bundle $E\to S$, consider the pull back of $E\to S$ along $E\to S$, i.e. the vector bundle $\mathcal{E}\to E$ fitting into the following Cartesian square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & E \arrow[d] \\ E \arrow[r] & S. \end{tikzcd}$$

The *ghost/ghost-momentum bundle* of $E\to S$ is the vector bundle

$$\wedge(\mathcal{E}\oplus\mathcal{E}^*)\to E.$$

We denote the set of sections of the ghost/ghost-momentum bundle by $BFV(E)$.

REMARK 2.3. The ghost/ghost-momentum bundle is of the form treated in the previous Section. In particular all the structures described in Remark 1.1 are present. We adopt the notation introduced in Remark 1.1. In particular $BFV^{(p,q)}(E)$ is the graded vector space of sections of ghost degree p and ghost-momentum degree q, $BFV^k(E)$ is the graded vector space of sections of total degree k (recall that the total degree is the ghost degree minus the ghost-momentum degree) and $BFV_{\geq r}(E)$ is the ideal of elements with ghost momentum degree at least r. Furthermore we denote the projection

$$BFV(E)\to BFV^{(0,0)}(E)=\mathcal{C}^\infty(E)$$

by π.

Recall that $BFV(E)$ is equipped with a graded Poisson bracket $[\cdot,\cdot]_G$ encoding the pairing between $\mathcal{E}$ and $\mathcal{E}^*$.

COROLLARY 2.4. *Let $E \to S$ be a vector bundle whose total space E is equipped with a Poisson bivector field Π.*

There is a graded Poisson bracket $[\cdot,\cdot]_{BFV}$ on $BFV(E)$ of degree 0 such that

(a) *The restriction of $\pi \circ [\cdot,\cdot]_{BFV}$ to $\mathcal{C}^\infty(E) \times \mathcal{C}^\infty(E)$ coincides with the Poisson bracket $\{\cdot,\cdot\}_\Pi$ associated to Π.*
(b) *The restriction of $\pi \circ [\cdot,\cdot]_{BFV}$ to $\Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}^*)$ coincides with the pairing between $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{E}^*)$ induced by the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$.*

Moreover any two such brackets are equivalent up to an automorphism of graded algebras $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$.

PROOF. This is an immediate consequence of Corollary 1.6 and Proposition 1.8. □

DEFINITION 2.5. A *BFV-bracket* is a graded Poisson bracket of degree 0 on $BFV(E)$ satisfying properties (a) and (b) from Corollary 2.4.

REMARK 2.6. Let $E \xrightarrow{p} S$ be a vector bundle. We denote the pull back of $E \to S$ along $E \to S$ by $\mathcal{E} \to E$. By definition as a pull back bundle $\mathcal{E} \to E$ comes along with a surjective vector bundle morphism $\mathcal{E} \to E$ covering $E \to S$. It is easy to check that the pull back of $\mathcal{E} \to E$ along $i : S \hookrightarrow E$ is naturally isomorphic to $E \to S$, hence there is an injective vector bundle morphism $E \to \mathcal{E}$ covering $S \hookrightarrow E$.

Sections of $\mathcal{E} \to E$ are in one-to-one correspondence with maps $E \to E$ making the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, dashed] \arrow[r, "p"] & S \end{tikzcd}$$

commutative. Observe that $\mathcal{E} \to E$ comes along with a *tautological section* Ω^0 that corresponds to $E \xrightarrow{\mathrm{id}} E$. Furthermore there is a natural map $p^* : \Gamma(E) \to \Gamma(\mathcal{E})$ given by mapping a section $s : S \to E$ to $E \xrightarrow{p} S \xrightarrow{s} E$. The map $S \hookrightarrow E$ induces a map $i^* : \Gamma(\mathcal{E}) \to \Gamma(E)$ in the same manner.

LEMMA 2.7. *Let $E \to S$ be a vector bundle and consider $BFV(E)$.*

The linear map of degree $+1$

$$\delta := [\Omega^0, \cdot]_G$$

is a differential on $BFV(E)$, i.e. it is a graded derivation of degree 1 and a coboundary operator.

Moreover the cohomology $H(BFV(E), \delta)$ is isomorphic to the algebra $\Gamma(\wedge E)$.

Proof. The graded derivation property follows from the graded Jacobi identity of $[\cdot,\cdot]_G$. The identity $\delta\circ\delta = 0$ is a direct consequence of $[\Omega^0,\Omega^0]_G = 0$ which is true because G is given by the contraction between $\mathcal{E}$ and $\mathcal{E}^*$ and Ω^0 is an element of ghost-momentum degree 0.

The two maps $p^* : \Gamma(E) \to \Gamma(\mathcal{E})$ and $i^* : \Gamma(\mathcal{E}) \to \Gamma(E)$ uniquely extend to

$$\begin{aligned}&\wedge p^* : \Gamma(\wedge E) \to \Gamma(\wedge\mathcal{E}) \hookrightarrow \Gamma(\wedge(\mathcal{E}\oplus\mathcal{E}^*)) \text{ and}\\ &\wedge i^* : \Gamma(\wedge(\mathcal{E}\oplus\mathcal{E}^*)) \to \Gamma(\wedge\mathcal{E}) \to \Gamma(\wedge E).\end{aligned}$$

The relations $\wedge i^*\circ\wedge p^* = \mathrm{id}$, $\delta\circ\wedge p^* = 0$ and $\wedge i^*\circ\delta = 0$ are straightforward to check.

We introduce the following local coordinates on $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$: $(x^\beta)_{\beta=1,\dots,s}$ are local coordinates on S, $(y^i)_{i=1,\dots,e}$ are linear coordinates along the fibres of E, $(b^i)_{i=1,\dots,e}$ is the corresponding local frame on $\mathcal{E}^*$ and $(c_i)_{i=1,\dots,e}$ is the dual frame on $\mathcal{E}$. These two frames yield a frame of $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$. The tautological section Ω^0 of $\mathcal{E}\to E$ is locally given by

$$\sum_{i=1}^{e} y^i c_i.$$

Consequently the differential $\delta := [\Omega^0,\cdot]_G$ reads

$$\sum_{i=1}^{e} y^i \frac{\partial}{\partial b^i}.$$

Define $h : BFV(E) \to BFV(E)[-1]$ locally by

$$h(f(x,y,c)b^{i_1}\cdots b^{i_k}) := \sum_{j=1}^{e} b^j \left(\int_0^1 \frac{\partial f}{\partial y^j}(x, t\cdot y, c)t^k\right) b^{i_1}\cdots b^{i_k}.$$

Since $(b^i)_{i=1,\dots,e}$ transforms dually to $(\frac{\partial}{\partial y^i})_{i=1,\dots,e}$ the operator h is globally well-defined. The identities $h\circ h = 0$, $\wedge i^*\circ h = 0$ and $h\circ\wedge p^* = 0$ are easily verified. We claim that

$$[h,\delta] = h\circ\delta + \delta\circ h = \mathrm{id} - (\wedge p^*)\circ(\wedge^* i)$$

holds. The elementary but tedious verification of this identity is done in Lemma 7 in Chapter 6. It implies that $\wedge p^*$ and $\wedge i^*$ induce inverse maps between $H(BFV(E),\delta)$ and $H(\Gamma(\wedge E), 0) = \Gamma(\wedge E)$. □

Remark 2.8. The proof of Lemma 2.7 yields a criterion whether a given cocycle of $(BFV(E),\delta)$ is a coboundary: a cocycle whose component of ghost-momentum degree 0 vanishes when restriction to S is a coboundary.

Definition 2.9. A *coisotropic vector bundle* is a pair $(E\to S,\Pi)$ where $E\to S$ is a vector bundle and Π is a Poisson bivector field on E such that the zero section S is a coisotropic submanifold of (E,Π).

REMARK 2.10. Let Ω be an element of $BFV^1(E)$. By definition

$$BFV^1(E) = \oplus_{k\geq 0}\Gamma(\wedge^{k+1}\mathcal{E} \otimes \wedge^k \mathcal{E}^*)$$

and consequently Ω decomposes into the sum of components Ω^k in $\Gamma(\wedge^{k+1}\mathcal{E}\otimes\wedge^k\mathcal{E}^*)$ for $k \geq 0$.

LEMMA 2.11. *Let $E \to S$ be a vector bundle whose total space E is equipped with a Poisson bivector field Π. Fix a connection ∇ on the pull back bundle $\mathcal{E} \to E$ and consider the biderivation $[\cdot,\cdot]_{i_\nabla(\Pi)}$ corresponding to the horizontal lift of Π to $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$.*

Then the following statements are equivalent:

(a) (E, Π) *is a coisotropic vector bundle.*
(b) *The restriction of $[\Omega^0, \Omega^0]_{i_\nabla \Pi}$ to S vanishes.*

PROOF. Let $(x^\beta)_{\beta=1,\dots,s}$ be a system of local coordinates on S, $(y^i)_{i=1,\dots,e}$ linear coordinates along E, $(b^i)_{i=1,\dots,e}$ the corresponding local frame on $\mathcal{E}^*$ and $(c_i)_{i=1,\dots,e}$ the dual frame on $\mathcal{E}$. These two frames yield a frame of $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$. We compute locally

$$[\Omega^0, \Omega^0]_{i_\nabla(\Pi)} = \sum_{i,j=1}^{e} \left(y^i y^j [c_i, c_j]_{i_\nabla(\Pi)} + 2y^i [c_i, y^j]_{i_\nabla(\Pi)} c_j + [y^i, y^j]_{i_\nabla(\Pi)} c_i c_j\right).$$

and consequently the restriction of $[\Omega^0, \Omega^0]_{i_\nabla(\Pi)}$ to S vanishes if and only if the restriction of

$$[y^i, y^j]_{i_\nabla(\Pi)}$$

to S vanishes. Because $(y^i)_{i=1,\dots,e}$ are linear fibre coordinates along $E \to S$ and the horizontal lift i_∇ is taken with respect to a pull back connection, $[y^i, y^j]_{i_\nabla(\Pi)}$ is equal to $\{y^i, y^j\}_\Pi$ and hence the restriction of $[\Omega^0, \Omega^0]_{i_\nabla(\Pi)}$ to S vanishes if and only if the restriction of

$$\{y^i, y^j\}_\Pi$$

to S vanishes. Lemma 8 in Chapter 6 asserts that $(y^i)_{i=1,\dots,e}$ is a local system of generators of the vanishing ideal $\mathcal{I}_S$ of S in E, i.e. any locally defined smooth functions h and g in $\mathcal{I}_S$ can be written as

$$h(x, y) = \sum_{i=1}^{e} h_i(x, y) y^i, \quad g(x, y) = \sum_{i=1}^{e} g_i(x, y) y^i$$

and the graded derivation property for $\{\cdot,\cdot\}_\Pi$ implies

$$\{h, g\}_\Pi = \sum_{i,j=1}^{e} \left(\{h_i, g_j\}_\Pi y^i y^j + \{h_i, y^j\}_\Pi y^i g_j + h_i \{y^i, g_j\}_\Pi y^j + h_i g_j \{y^i, y^j\}_\Pi\right).$$

Consequently the restriction of $\{h, g\}_\Pi$ to S vanishes if and only the restriction of $\{y^i, y^j\}_\Pi$ to S vanishes.

Hence the restriction of $[\Omega^0, \Omega^0]_{i_\nabla(\Pi)}$ to S vanishes if and only if the vanishing ideal $\mathcal{I}_S$ of S in E is locally closed under $\{\cdot,\cdot\}_\Pi$. By Lemma 2.3 and Remark 2.4 in Chapter 3 the local closedness of $\mathcal{I}_S$ under $\{\cdot,\cdot\}_\Pi$ is equivalent to S being a coisotropic submanifold of (E, Π). □

DEFINITION 2.12. Let $E \to S$ be a vector bundle whose total space E is equipped with a Poisson bivector field Π. Fix a BFV-bracket $[\cdot,\cdot]_{BFV}$ on $BFV(E)$. A *BFV-charge* of $(BFV(E), [\cdot,\cdot]_{BFV})$ is an element of $BFV^1(E)$ such that

(i) $[\Omega, \Omega]_{BFV} = 0$ and
(ii) the component of Ω in $\Gamma(\mathcal{E})$ is the tautological section Ω^0 of $\mathcal{E} \to E$.

THEOREM 2.13. *Let $E \to S$ be a vector bundle whose total space E is equipped with a Poisson bivector field Π. Fix a BFV-bracket $[\cdot,\cdot]_{BFV}$ on $BFV(E)$.*

Then the following statements are equivalent:

(a) *(E, Π) is a coisotropic vector bundle.*
(b) *A BFV-charge Ω of $(BFV(E), [\cdot,\cdot]_{BFV})$ exists.*

Moreover given two BFV-charges Ω and Ω' of $(BFV(E), [\cdot,\cdot]_{BFV})$ there is an automorphism Ψ of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ that maps Ω to Ω'.

REMARK 2.14. The proof we gave essentially follows [**Sta2**]. Some adaptations to the smooth setting were presented in [**He**].

PROOF. Consider the claimed equivalence $(a) \Leftrightarrow (b)$. Observe that due to Proposition 1.8 this statement does not depend on the particular BFV-bracket we choose. So let us pick a BFV-bracket $[\cdot,\cdot]_{BFV}$ constructed with the help of the L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ from Proposition 1.4.

First we show the implication $(b) \Rightarrow (a)$, i.e. we claim that the existence of $\Omega \in BFV^1(E)$ satisfying (i) and (ii) implies that the zero section S of $E \to S$ is a coisotropic submanifold. Observe that $[\Omega, \Omega]_{BFV}$ is an element of $BFV^2(E)$ and hence decomposes with respect to

$$BFV^2(E) = \oplus_{k \geq 0} \Gamma(\wedge^{k+2}\mathcal{E} \otimes \wedge^k \mathcal{E}^*).$$

We want to compute the component of $[\Omega, \Omega]_{BFV}$ in $\Gamma(\wedge^2\mathcal{E} \otimes \wedge^0\mathcal{E}^*)$. The BFV-bracket $[\cdot,\cdot]_{BFV}$ decomposes into

$$[\cdot,\cdot]_{BFV} = [\cdot,\cdot]_G + [\cdot,\cdot]_{i_\nabla(\Pi)} + \cdots$$

where $\cdots$ refers to terms that increase the ghost-momentum degree by at least 1 and $[\cdot,\cdot]_{i_\nabla(\Pi)}$ is the biderivation associated to the horizontal lift of Π to a bivector on $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$. We compute

$$\begin{aligned}
[\Omega,\Omega]_{BFV} &= [\Omega^0+\Omega^1,\Omega^0+\Omega^1]_{BFV} + BFV_{\geq 1}(E) \\
&= [\Omega^0+\Omega^1,\Omega^0+\Omega^1]_G + [\Omega^0+\Omega^1,\Omega^0+\Omega^1]_{i_\nabla(\Pi)} + BFV_{\geq 1}(E) \\
&= 2[\Omega^0,\Omega^1]_G + [\Omega^0,\Omega^0]_{i_\nabla(\Pi)} + BFV_{\geq 1}(E) \\
&= 2\delta(\Omega^1) + [\Omega^0,\Omega^0]_{i_\nabla(\Pi)} + BFV_{\geq 1}(E)
\end{aligned}$$

and hence $[\Omega,\Omega]_{BFV} = 0$ implies

$$2\delta(\Omega^1) + [\Omega^0,\Omega^0]_{i_\nabla(\Pi)} = 0.$$

Because the restriction of something in the image of δ to S always vanishes, the above identity implies that the restriction of $[\Omega^0,\Omega^0]_{i_\nabla(\Pi)}$ to S also vanishes. By Lemma 2.11 this means that S is a coisotropic submanifold.

On the other hand, suppose that S is a coisotropic submanifold of (E,Π). Our first aim is to find $\Omega^1 \in BFV^{(2,1)}(E)$ such that

$$[\Omega^0+\Omega^1,\Omega^0+\Omega^1]_{BFV} = 0 + BFV_{\geq 1}(E)$$

holds. The calculation from above shows that this is equivalent to finding Ω^1 that satisfies

$$2\delta(\Omega^1) + [\Omega^0,\Omega^0]_{i_\nabla(\Pi)} = 0.$$

Because $R(0) := -1/2[\Omega^0,\Omega^0]_{i_\nabla(\Pi)}$ is an element of $BFV^{(2,0)}(E)$, $R(0)$ is closed with respect to δ. Moreover the cohomology class of $R(0)$ vanishes: the class is an element of $\Gamma(\wedge^2 E)$ obtained be restricting $R(0)$ to S. However S is a coisotropic submanifold and by Lemma 2.11 this is equivalent to $-2R(0)|_S = 0$. Hence we can find $\Omega^1 \in BFV^{(2,1)}(E)$ satisfying

$$\delta(\Omega^1) = R(0) = -\frac{1}{2}[\Omega^0,\Omega^0]_{i_\nabla(\Pi)}.$$

A possible choice of Ω^1 is $h(R(0))$ where h is the homotopy introduced in the proof of Lemma 2.7.

Assume that we found $\Omega(k) := \Omega^0 + \cdots + \Omega^k$ for $k > 0$ with $\Omega^k \in BFV^{(k+1,k)}(E)$ satisfying

$$[\Omega(k),\Omega(k)]_{BFV} = 0 + BFV_{\geq k}(E).$$

We want to find $\Omega^{k+1} \in BFV^{(k+2,k+1)}(E)$ such that

$$[\Omega(k)+\Omega^{k+1},\Omega(k)+\Omega^{k+1}]_{BFV} = 0 + BFV_{\geq (k+1)}(E)$$

holds. Set

$$R(k) := -\frac{1}{2}[\Omega(k),\Omega(k)]_{BFV} + BFV_{\geq (k+1)}(E),$$

i.e. $R(k) \in BFV^{(k+2,k)}(E)$. The graded Jacobi identity for $[\cdot,\cdot]_{BFV}$ implies

$$[\Omega(k),[\Omega(k),\Omega(k)]_{BFV}]_{BFV} = 0$$

and hence

$$\begin{aligned}-2\delta(R(k)) &= [\Omega^0, -2R(k)]_{BFV} + BFV_{\geq k}(E)\\ &= [\Omega(k), [\Omega(k), \Omega(k)]_{BFV}]_{BFV} + BFV_{\geq k}(E) = 0.\end{aligned}$$

So $R(k)$ is closed with respect to δ and because it is concentrated in positive ghost-momentum degrees its cohomology class vanishes (Remark 2.8). Therefore we can find $\Omega^{(k+1)} \in BFV^{(k+2,k+1)}(E)$ satisfying

$$\delta(\Omega^{(k+1)}) = R(k)$$

and obtain

$$\begin{aligned}&[\Omega(k) + \Omega^{(k+1)}, \Omega(k) + \Omega^{(k+1)}]_{BFV} + BFV_{\geq(k+1)}(E) =\\ &= [\Omega(k), \Omega(k)]_{BFV} + 2[\Omega(k), \Omega^{(k+1)}]_{BFV} + BFV_{\geq(k+1)}(E)\\ &= -2R(k) + \delta(\Omega^{(k+1)}) + BFV_{\geq(k+1)}(E)\\ &= 0 + BFV_{\geq(k+1)}(E).\end{aligned}$$

The filtration $BFV_{\geq r}(E)$ is bounded from above so we can consecutively find a finite number of appropriate correction terms Ω^k such that $\Omega := \Omega^0 + \Omega^1 + \cdots + \Omega^N$ satisfies

$$[\Omega, \Omega]_{BFV} = 0.$$

Finally let Ω and Ω' be two elements of $BFV^1(E)$ satisfying properties (i) and (ii) stated in Definition 2.12. Assume $\gamma := \Omega - \Omega'$ lies in $BFV_{\geq k}(E)$ for some $k > 0$. Denote its component in $BFV^{(k+1,k)}(E)$ by γ^k. We obtain

$$0 = [\Omega, \Omega]_{BFV} + BFV_{\geq(k-1)}(E) = 2\delta(\Omega^k) + F(\Omega^1, \dots, \Omega^{(k-1)})$$

where F is some quadratic term depending on $(\Omega^1, \cdots, \Omega^{(k-1)})$ only. Because of

$$\begin{aligned}\delta(\gamma^k) &= \delta(\Omega^k - \Omega'^k)\\ &= -\frac{1}{2}\left(F(\Omega^1, \cdots, \Omega^{(k-1)}) - F(\Omega'^1, \cdots, \Omega'^{(k-1)})\right) = 0\end{aligned}$$

the element γ^k is a cocycle of $(BFV(E), \delta)$. By Lemma 2.7 and Remark 2.8 it is a coboundary, i.e. we can find $\varepsilon^k \in BFV^{(k+1,k+1)}(E)$ with $\delta(\varepsilon^k) = \gamma^k$.

Since $k > 0$, $[\varepsilon_k, \cdot]_{BFV}$ acts as a nilpotent inner derivation on the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ and integrates to an automorphism $\Phi(k)$ of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$. We calculate

$$\begin{aligned}\Phi(k)(\Omega) + BFV_{\geq(k+1)}(E) &= \Omega + [\varepsilon^k, \Omega]_{BFV} + BFV_{\geq(k+1)}(E)\\ &= \Omega^1 + \cdots + \Omega^k + [\varepsilon^k, \Omega^0]_G + BFV_{\geq(k+1)}(E)\\ &= \Omega^1 + \cdots + \Omega^k - \delta(\varepsilon^k) + BFV_{\geq(k+1)}(E)\\ &= \Omega^1 + \cdots + \Omega^{(k-1)} + \Omega'^k + BFV_{\geq(k+1)}(E)\\ &= \Omega' + BFV_{\geq(k+1)}(E).\end{aligned}$$

This implies that $\Phi(\Omega) - \Omega'$ lies in $BFV_{\geq(k+1)}(E)$.

The filtration $BFV_{\geq(k+1)}(E)$ is bounded from above and we can consecutively find automorphisms $\Phi(1),\ldots,\Phi(N)$ of the graded Poisson algebra $(BFV(E),[\cdot,\cdot]_{BFV})$ such that $\Phi(N)\circ\cdots\circ\Phi(1)$ maps Ω to Ω'. □

Definition 2.15. Let (E,Π) be a coisotropic vector bundle. A *BFV-complex* associated to (E,Π) is a choice of a BFV-bracket $[\cdot,\cdot]_{BFV}$ on $BFV(E)$ and of a BFV-charge Ω of $(BFV(E),[\cdot,\cdot]_{BFV})$.

Corollary 2.16. *Associated to any BFV-complex $(BFV(E),[\cdot,\cdot]_{BFV},\Omega)$ is a differential graded Poisson algebra*

$$(BFV(E),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV}).$$

The isomorphism type of this differential graded Poisson algebra is independent of the specific choice of a BFV-bracket $[\cdot,\cdot]_{BFV}$ on $BFV(E)$ and of a BFV-charge Ω of $(BFV(E),[\cdot,\cdot]_{BFV})$.

Proof. Definition 2.12, Proposition 1.8 and Theorem 2.13 immediately imply the Lemma. □

Remark 2.17. By Corollary 2.16 the isomorphism type of the differential graded Poisson algebra $(BFV(E),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV})$ is an invariant of the underlying coisotropic vector bundle (E,Π). In Section 3 this differential graded Poisson algebra will turn out to be tightly related to the L_∞-algebra structure on $\Gamma(\wedge E)$ introduced in Corollary 3.4, Chapter 3. Moreover it has remarkable connections to the deformation problem of coisotropic submanifolds on which we will elaborate in Chapter 5.

Definition 2.18. Let S be a coisotropic submanifold of a Poisson manifold (M,Π) and σ an embedding of the normal bundle NS of S in M into M whose restriction to S is the identity. A *BFV-complex* associated to (S,σ) is a BFV-complex associated to the coisotropic vector bundle (NS,Π_σ) where Π_σ denotes the Poisson bivector field that NS inherits from (M,Π) via the identification $NS\cong\sigma(NS)\subset M$.

Lemma 2.19. *Given a BFV-complex $(BFV(NS),\Omega,[\cdot,\cdot]_{BFV})$ associated to (S,σ) where S is a coisotropic submanifold of the Poisson manifold (M,Π), the cohomology of*

$$(BFV(NS),[\Omega,\cdot]_{BFV})$$

is isomorphic to the Lie algebroid cohomology of S in (M,Π) – see Definition 2.13 in Chapter 3.

Proof. Set $E:=NS\to S$. By Corollary 2.16 we can assume without loss of generality that the BFV-bracket $[\cdot,\cdot]_{BFV}$ under consideration is given in terms of a L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ introduced in Proposition 1.4. Moreover the component

Ω^1 of the BFV-charge Ω of $(BFV(E), [\cdot,\cdot]_{BFV})$ in $BFV^{(2,1)}(E)$ can be assumed to be equal to

$$-\frac{1}{2}h([\Omega^0,\Omega^0]_{i_\nabla(\Pi)}),$$

see the proof of Theorem 2.13.

The differential $[\Omega,\cdot]_{BFV}$ has total degree one and maps the ideal $BFV_{\geq r}(E)$ to $BFV_{\geq(r-1)}(E)$. Consequently it preserves the ideals $\Gamma(\wedge^{\geq s}\mathcal{E}\otimes\wedge\mathcal{E}^*)$. This family of ideals yields a bounded filtration of $BFV(E)$. We want to compute the spectral sequence associated to this filtration.

The first sheet E_1 is given by the direct sum of complexes

$$\Gamma(\wedge^r\mathcal{E}\otimes\mathcal{E}^*)\cong\Gamma(\wedge^{\geq r}\mathcal{E}\otimes\wedge\mathcal{E}^*)/\Gamma(\wedge^{\geq(r+1)}\mathcal{E}\otimes\wedge\mathcal{E}^*)$$

with differential d_1 defined on $X\in\Gamma(\wedge^s\mathcal{E}\otimes\mathcal{E}^*)$ by

$$\begin{aligned} d_1(X) &= [\Omega,X]_{BFV}+\Gamma(\wedge^{(s+1)}\mathcal{E}\otimes\mathcal{E}^*)\\ &= [\Omega^0,X]_G+\Gamma(\wedge^{(s+1)}\mathcal{E}\otimes\mathcal{E}^*)\\ &= \delta(X)+\Gamma(\wedge^{(s+1)}\mathcal{E}\otimes\mathcal{E}^*),\end{aligned}$$

i.e. the first sheet is isomorphic to $(BFV(E),\delta)$. By Lemma 2.7 the cohomology of $(BFV(E),\delta)$ is isomorphic to the graded algebra $\Gamma(\wedge E)$ which is the second sheet E_2 of the spectral sequence. Moreover this means that the spectral sequence under consideration collapses after one step. Thus the cohomology of $(BFV(E),[\Omega,\cdot]_{BFV})$ is isomorphic to the cohomology of $(E_2,d_2)\cong E_\infty$.

Next we calculate the image of $Y\in\Gamma(\wedge^s E)$ under the induced differential d_2 on $E_2=\Gamma(\wedge E)$. Recall that there are morphism of complexes

$$\wedge p^*:(\Gamma(\wedge E),0)\to(BFV(E),\delta),\quad \wedge i^*:(BFV(E),\delta)\to(\Gamma(\wedge E),0)$$

which were introduced in the proof of Lemma 2.7. We compute

$$\begin{aligned} d_2(Y) &= \wedge i^*\left([\Omega,\wedge p^*(Y)]_{BFV}+\Gamma(\wedge^{(s+2)}\mathcal{E}\otimes\wedge\mathcal{E}^*)\right)\\ &= \wedge i^*\left([\Omega^0,\wedge p^*(Y)]_G+[\Omega^0,\wedge p^*(Y)]_{i_\nabla(\Pi)}+[\Omega^1,\wedge p^*(Y)]_G+\Gamma(\wedge^{(s+2)}\mathcal{E}\otimes\wedge\mathcal{E}^*)\right)\\ &= \wedge i^*\left([\Omega^0,\wedge p^*(Y)]_{i_\nabla(\Pi)}-\frac{1}{2}[h([\Omega^0,\Omega^0]_{i_\nabla(\Pi)}),\wedge p^*(Y)]_G+\Gamma(\wedge^{(s+2)}\mathcal{E}\otimes\wedge\mathcal{E}^*)\right).\end{aligned}$$

Since $\Gamma(\wedge E)$ is locally generated by $\mathcal{C}^\infty(S)$ and $\Gamma(E)$, it suffices to know d_2 on $\mathcal{C}^\infty(S)\oplus\Gamma(E)$. For f a smooth function on S we obtain

$$d_2(f)=([\Omega^0,p^*(f)]_{i_\nabla(\Pi)})|_S.$$

Contraction with an arbitrary $\lambda\in\Gamma(N^*S)$ yields

$$\begin{aligned} <d_2(f),\lambda> &= <[\Omega^0,p^*(f)]_{i_\nabla(\Pi)})|_S,\lambda>\\ &= <\Pi^\#|_S(\lambda),d_{DR}f>\\ &= <P([\Pi,p^*(f)]_{SN}),\lambda>\\ &= <\partial_\Pi(f),\lambda>\end{aligned}$$

hence $d_2(f) = \partial_\Pi(f)$. We made use of the definition of the tautological section Ω^0 and of the identity

$$< P([\Pi, \tilde{f}]_{SN}), \lambda > = < \Pi^\#|_S(\lambda), d_{DR} f > = \left(\Pi^\#|_S(\lambda)\right)(f)$$

which was established in the proof of part (d) of Lemma 2.12 in Chapter 3.

Let $(x^\beta)_{\beta=1,\dots,s}$ be local coordinates on S, $(y^i)_{i=1,\dots,e}$ linear coordinates along the fibres of E, $(b^i)_{i=1,\dots,e}$ the corresponding local frame on $\mathcal{E}^*$ and $(c_i)_{i=1,\dots,e}$ the dual frame on $\mathcal{E}$. The tautological section Ω^0 of $\mathcal{E} \to E$ is locally given by

$$\sum_{i=1}^{e} y^i c_i.$$

and the homotopy $h : BFV(E) \to BFV(E)[-1]$ by

$$h(f(x,y,c) b^{i_1} \cdots b^{i_k}) := \sum_{j=1}^{e} b^j \left(\int_0^1 \frac{\partial f}{\partial y^j}(x, t \cdot y, c) t^k \right) b^{i_1} \cdots b^{i_k}.$$

We compute

$$\begin{aligned}
d_2(i^*(c_k)) &= \wedge i^* \left([\Omega^0, c_k]_{i_\nabla(\Pi)} - \frac{1}{2}[h([\Omega^0, \Omega^0]_{i_\nabla(\Pi)}), c_k]_G \right) \\
&= [\Omega^0, c_k]_{i_\nabla(\Pi)}|_S - \frac{1}{2}\left(< h([\Omega^0, \Omega^0]_{i_\nabla(\Pi)}), c_k >\right)|_S \\
&= \left(\sum_{l=1}^{e} c_l [y^l, c_k]_{i_\nabla(\Pi)} - \frac{1}{2} \int_0^1 \frac{\partial}{\partial y^k}\left([\Omega^0, \Omega^0]_{i_\nabla(\Pi)}\right)(x, t \cdot y, c) dt \right)|_S \\
&= \left(\sum_{l=1}^{e} c_l [y^l, c_k]_{i_\nabla(\Pi)} \right)|_S - \frac{1}{2} \int_0^1 \frac{\partial}{\partial y^k}\left([\Omega^0, \Omega^0]_{i_\nabla(\Pi)}\right)(x, 0, c) dt \\
&= \left(\sum_{l=1}^{e} c_l [y^l, c_k]_{i_\nabla(\Pi)} \right)|_S - \frac{1}{2} \frac{\partial}{\partial y^k}\left([\Omega^0, \Omega^0]_{i_\nabla(\Pi)}\right)(x, 0, c) \\
&= \left(\sum_{l=1}^{e} c_l [y^l, c_k]_{i_\nabla(\Pi)} \right)|_S \\
&\quad - \frac{1}{2} \sum_{i,j=1}^{e} \frac{\partial}{\partial y^k}\left(y^i [c_i, c_j]_{i_\nabla(\Pi)} y^j + 2 y^i [c_i, y^j]_{i_\nabla(\Pi)} c_j + c_i c_j [y^i, y^j]_{i_\nabla(\Pi)} \right)|_S \\
&= \left(\sum_{l=1}^{e} c_l [y^l, c_k]_{i_\nabla(\Pi)} \right)|_S - \left(\sum_{j=1}^{e} [c_k, y^j]_{i_\nabla(\Pi)} c_j \right)|_S \\
&\quad - \left(\frac{1}{2} \sum_{i,j=1}^{e} c_i c_j \frac{\partial}{\partial y^k}\left([y^i, y^j]_{i_\nabla(\Pi)}\right) \right)|_S \\
&= -\frac{1}{2} \sum_{i,j=1}^{e} c_i c_j \frac{\partial}{\partial y^k}\left(\{y^i, y^j\}_\Pi\right)|_S
\end{aligned}$$

which is the local expression of

$$\wedge\,\mathrm{pr}\,(-[c_k,\Pi]_{SN}|_S) = \wedge\,\mathrm{pr}\,([\Pi,c_k]|_S) = \partial_\Pi(i^*(c_k)).$$

$\square$

LEMMA 2.20. *Let $(BFV(NS),[\cdot,\cdot]_{BFV},\Omega)$ be a BFV-complex associated to (S,σ) where S is a coisotropic submanifold of the Poisson manifold (M,Π).*

The graded Poisson bracket $[\cdot,\cdot]_{BFV}$ induces a Poisson bracket on the zero'th cohomology $H^0(BFV(E),[\Omega,\cdot]_{BFV})$ which is isomorphic to the Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$ on the quotient algebra $\mathcal{A}(\underline{S})$ – see Lemma 2.22 in Chapter 3 – under the identification

$$H^0(BFV(NS),[\Omega,\cdot]_{BFV}) \cong H^0(\Gamma(\wedge NS),\partial_\Pi) \cong \mathcal{A}(\underline{S}).$$

PROOF. The verification that $[\cdot,\cdot]_{BFV}$ induces a Poisson bracket on the zero'th cohomology of $(BFV(NS),[\Omega,\cdot]_{BFV})$ can be copied mutatis mutandis from the corresponding verification in the proof of part (b)(ii) of Lemma 3.7 in Chapter 3.

The isomorphism

$$H(BFV(NS),[\Omega,\cdot]_{BFV}) \cong E_\infty \cong H_{d_2}(E_2) = H(\Gamma(\wedge NS),\partial_\Pi)$$

established in the proof of Lemma 2.19 implies that every cohomology class of degree 0 of $(BFV(NS),[\Omega,\cdot]_{BFV})$ can be represented by a cocycle of the form

$$F = p^*(f) + BFV_{\geq 1}(NS)$$

where f is a function on S satisfying $\partial_\Pi(f) = 0$ and $p^*(f)$ is its pull back to a function on $E \to S$. The isomorphism between $H^0(BFV(NS),[\Omega,\cdot]_{BFV})$ and $H^0(\Gamma(\wedge NS),\partial_\Pi)$ is given by

$$[F = p^*(f) + BFV_{\geq 1}(NS)] \mapsto f \in \mathcal{C}^\infty(S).$$

Let F and G be two such cocycles representing cohomology classes in degree 0 with associated functions f and g on S. We compute

$$\begin{aligned} [F,G]_{BFV} &= [p^*(f),p^*(g)]_{BFV} + BFV_{\geq 1}(NS) \\ &= \{p^*(f),p^*(g)\}_{\Pi_\sigma} + BFV_{\geq 1}(NS) \end{aligned}$$

Consequently the class of $[F,G]_{BFV}$ in

$$BFV^0(BFV(NS),[\Omega,\cdot]_{BFV}) \cong \mathcal{A}(\underline{S})$$

is $\{f,g\}_{\underline{\Pi}}$, i.e. the induced Poisson bracket on $H(BFV(NS)[\Omega,\cdot]_{BFV})$ is isomorphic to $\{\cdot,\cdot\}_{\underline{\Pi}}$. $\square$

REMARK 2.21. Let S be a coisotropic submanifold of a Poisson manifold (M,Π). By Lemma 2.19, Lemma 2.20, and Lemma 3.7 in Chapter 3 both the differential graded Poisson algebra

$$(BFV(NS),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV})$$

and the P_∞-algebra

$$(\Gamma(\wedge NS), \lambda_n^\sigma)$$

provide some sort of "resolution" of the Poisson algebra $(\mathcal{A}(\underline{S}), \{\cdot,\cdot\}_{\underline{\Pi}})$ introduced in Section 2, Chapter 3. More precisely both are chain complexes whose zero'th cohomology is isomorphic to the algebra $\mathcal{A}(\underline{S})$ and the Poisson bracket $\{\cdot,\cdot\}_{\underline{\Pi}}$ is induced from some structure defined on the two complexes. Observe that both complexes are in general not acyclic in positive degrees but their cohomology is isomorphic by Lemma 2.19. This observation will be extended considerably in the next Section, see Theorem 3.6 in particular.

Although the differential graded Poisson algebra associated to a BFV-complex and the homotopy Lie algebroid are tightly related there are some subtle differences which will play an important role in Chapter 5. In particular it turns out that the differential graded Poisson algebras associated to two BFV-complexes that correspond to different choices of embeddings of the normal bundle NS into M might not be isomorphic. In contrast, Theorem 3.15 in Chapter 3 asserts that the isomorphism type of the homotopy Lie algebroids associated to different choices of embeddings is always the same.

EXAMPLE 2.22. Consider the submanifold $S = \{0\}$ of $M = \mathbb{R}^2$ equipped with the Poisson bivector field

$$\Pi(x,y) := \begin{cases} 0 & x^2+y^2 \leq 4 \\ \exp\left(-\frac{1}{x^2+y^2-4}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} & x^2+y^2 \geq 4 \end{cases}.$$

Let σ_0 be the embedding $NS = \mathbb{R}^2 \to \mathbb{R}^2$ given by the identity and σ_1 the embedding given by

$$\mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \frac{1}{\sqrt{1+x^2+y^2}}(x,y).$$

The image of σ_1 is contained in the disk centered at $\{0\}$ with radius 1. Hence the Poisson bivector field Π_{σ_1} on $\mathbb{R}^2$ inherited from $(\mathbb{R}^2, \Pi)$ via σ_1 vanishes whereas $\Pi_{\sigma_0} = \Pi$ is not zero everywhere.

The ghost/ghost-momentum bundle of $\mathbb{R}^2 \to \{0\}$ is

$$\mathbb{R}^2 \oplus \wedge(\mathbb{R}^2 \oplus (\mathbb{R}^2)^*) \to \mathbb{R}^2.$$

Denote the graded Poisson bracket on the sections of the ghost/ghost-momentum bundle which encodes the natural pairing between $\mathbb{R}$ and $\mathbb{R}^*$ by $[\cdot,\cdot]_G$. A possible choice of a BFV-bracket for $(\mathbb{R}^2, \Pi)$ and $(\mathbb{R}^2, 0)$ is

$$[\cdot,\cdot]_G + \{\cdot,\cdot\}_\Pi \quad \text{and} \quad [\cdot,\cdot]_G \quad \text{respectively.}$$

Any isomorphism of graded Poisson algebras between

$$(BFV(\mathbb{R}^2), [\cdot,\cdot]_G + \{\cdot,\cdot\}_\Pi)$$

and

$$(BFV(\mathbb{R}^2), [\cdot, \cdot]_G)$$

would induce an isomorphism of Poisson algebras

$$(\mathcal{C}^\infty(\mathbb{R}^2), \{\cdot, \cdot\}_\Pi) \xrightarrow{\cong} (\mathcal{C}^\infty(\mathbb{R}^2), 0).$$

Such an isomorphism does not exist.

Although different choices of embeddings of the normal bundle can lead to differential graded Poisson algebras that are not isomorphic, it is always possible to find appropriate "restrictions" of the BFV-complexes under consideration such that the associated differential graded Poisson algebras are isomorphic.

Definition 2.23. Let $E \to S$ be a vector bundle and U an open neighbourhood of S in E which is equipped with a Poisson bivector field Π such that S is a coisotropic submanifold of (U, Π).

The *restricted ghost/ghost-momentum bundle* is the restriction of $\wedge(\mathcal{E} \oplus \mathcal{E}^*) \to E$ to U. Recall that $\mathcal{E} \to E$ denotes the pull back of $E \to S$ along $E \to S$. We set

$$BFV_U(E) := \Gamma(\wedge(\mathcal{E} \oplus \mathcal{E}^*)|_U).$$

A *restricted BFV-bracket* is a graded Poisson bracket $[\cdot, \cdot]_{BFV}$ of degree 0 on $BFV_U(E)$ such that

(a) The restriction of $\pi \circ [\cdot, \cdot]_{BFV}$ to $\mathcal{C}^\infty(U) \times \mathcal{C}^\infty(U)$ coincides with the Poisson bracket associated to Π.
(b) The restriction of $\pi \circ [\cdot, \cdot]_{BFV}$ to $\Gamma(\mathcal{E}|_U) \times \Gamma(\mathcal{E}^*|_U)$ coincides with the pairing between $\Gamma(\mathcal{E}|_U)$ and $\Gamma(\mathcal{E}^*|_U)$ induced by the natural fibre pairing between $\mathcal{E}|_U$ and $\mathcal{E}^*|_U$.

A *restricted BFV-charge* of $(BFV_U(E), [\cdot, \cdot]_{BFV})$ is an element of $BFV_U^1(E)$ such that

(i) $[\Omega, \Omega]_{BFV} = 0$ and
(ii) the component of Ω in $\Gamma(\mathcal{E}|_U)$ is the restriction of the tautological section Ω^0 of $\mathcal{E} \to E$ to U.

A *restricted BFV-complex* associated to $(E \to S, U, \Pi)$ is a choice of a BFV-bracket $[\cdot, \cdot]_{BFV}$ on $BFV_U(E)$ and a BFV-charge Ω of $(BFV_U(E), [\cdot, \cdot]_{BFV})$.

Clearly every BFV-complex $(BFV(E), [\cdot, \cdot]_{BFV}, \Omega)$ yields a restricted BFV-complex

$$(BFV(E), [\cdot, \cdot]_{BFV}, \Omega)|_U := (BFV_U(E), [\cdot, \cdot]_{BFV}, \Omega|_U)$$

which we refer to as the *restriction* of $(BFV(E), [\cdot, \cdot]_{BFV}, \Omega)$ to U.

Definition 2.24. Let $E \xrightarrow{p} S$ be a vector bundle. An open neighbourhood U of the zero section $S \xrightarrow{i} E$ is *contractible along fibres* if for all $x \in U$ the segment $[(i \circ p)(x), x]$ lies in U.

REMARK 2.25. Given an open neighbourhood V of the zero section S of some vector bundle $E \xrightarrow{p} S$ there is an open neighbourhood U of S in E which is contractible along fibres and that is contained in V. In fact, for every $s \in S$ fix a vector bundle chart and the restriction of V to this chart is an open neighbourhood of s. Consequently we can find an $\delta(s)$-ball $B_{\delta(s)}(s)$ centered at s which is contained in V. The union

$$U := \bigcup_{s \in S} B_{\delta(s)}(s)$$

is an open neighbourhood of S in E. Moreover for arbitrary $x \in U$ there is a $s \in S$ such that $x \in B_{\delta(s)}(s)$. This implies

$$[p(x), x] \subset B_{\delta(s)}(s) \subset U.$$

This proves that the set of open neighbourhood of S in E which are contractible along fibres is cofinal in the set of all open neighbourhoods of S in E seen as a partially ordered set with respect to inclusion.

LEMMA 2.26. *Let $E \to S$ be a vector bundle and U an open neighbourhood of S in E which is contractible along fibres. Moreover suppose Π is a Poisson bivector field Π on U such that S is a coisotropic submanifold of (U, Π).*

Restricted BFV-brackets on $BFV_U(E)$ exist and any two restricted BFV-brackets on $BFV_U(E)$ are related by an automorphism of the graded algebra $BFV_U(E)$.

BFV-charges of $(BFV_U(E), [\cdot,\cdot]_{BFV})$ exist and any two restricted BFV-charges of $(BFV_U(E), [\cdot,\cdot]_{BFV})$ are related by an automorphism of the graded Poisson algebra $(BFV_U(E), [\cdot,\cdot]_{BFV})$.

The isomorphism type of the differential graded Poisson algebra

$$(BFV_U(E), [\Omega, \cdot]_{BFV}, [\cdot,\cdot]_{BFV})$$

is independent of the specific choice of a restricted BFV-bracket $[\cdot,\cdot]_{BFV}$ on $BFV_U(E)$ and of a BFV-charge Ω of $(BFV_U(E), [\cdot,\cdot]_{BFV})$.

PROOF. The statement about the restricted BFV-brackets is an immediate consequence of Proposition 1.8.

The existence and uniqueness of restricted BFV-charges of $(BFV_U(E), [\cdot,\cdot]_{BFV})$ is proved in the same manner as Theorem 2.13. The crucial step is the computation of the cohomology of the complex

$$(BFV_U(E), \delta|_U(\cdot) := [\Omega^0|_U, \cdot]_G).$$

Because U is contractible along fibres the homotopy h introduced in Lemma 2.7 restricts to $BFV_U(E)$. This implies that any cocycle whose component in ghost-momentum degree 0 vanishes when restricted to S is a coboundary. This allows us to construct a BFV-charge by an inductive procedure as done in the proof of Theorem 2.13. Moreover the construction of an automorphism of the graded

Poisson algebra $(BFV_U(E), [\cdot,\cdot]_{BFV})$ relating two restricted BFV-charges can be copied mutatis mutandis from the proof of Theorem 2.13.

The statement about the restricted BFV-brackets and the restricted BFV-charges imply the invariance of the differential graded Poisson algebra associated to a restricted BFV-complex. □

REMARK 2.27. Given a coisotropic submanifold S of a Poisson manifold (M,Π) and a BFV-complex $(BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)$ associate to (S,σ), the restriction of $(BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)$ to an open neighbourhood U of S in NS yields a morphism of differential graded Poisson algebras

$$r|_U : (BFV(NS), [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV} \to (BFV_U(NS), [\Omega|_U,\cdot]_{BFV}, [\cdot,\cdot]_{BFV}).$$

If U is contractible along fibres, $r|_U$ is a quasi-isomorphism, i.e. it induces an isomorphism on cohomology. This follows from the computation of the cohomology $H(BFV_U(NS), [\Omega|_U,\cdot]_{BFV})$ in Lemma 2.26.

THEOREM 2.28. *Let S be a coisotropic submanifold of a Poisson manifold (M,Π). Suppose σ_0 and σ_1 are two embeddings of the normal bundle NS of S in M into M such that their restrictions to S are equal to* id_S.

Given BFV-complexes $(BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)$ and $(BFV(NS), [\cdot,\cdot]'_{BFV}, \Omega')$ associated to (S,σ_0) and (S,σ_1) respectively, there are

(a) *two open neighbourhoods U and V of S in E and*
(b) *an isomorphism of graded Poisson algebras*

$$\Phi : (BFV(NS), [\cdot,\cdot]_{BFV})|_U \xrightarrow{\cong} (BFV(NS), [\cdot,\cdot]'_{BFV})|_V$$

that maps $\Omega|_U$ to $\Omega|_V$.

PROOF. By Lemma 2.26 it suffices to find open neighbourhoods U and V of S in E and and an isomorphism of graded algebras

$$\Phi : BFV(NS)|_U \to BFV(NS)|_V$$

such that

$$\Phi \cdot (BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)|_U := (BFV_V(NS), \Phi([\Phi^{-1}(\cdot), \Phi^{-1}(\cdot)]_{BFV}), \Phi(\Omega|_U))$$

is a BFV-complex associated to (S,σ_1). By Proposition 1.8 and Corollary 2.16 we can assume without loss of generality that $[\cdot,\cdot]_{BFV}$ is a BFV-bracket constructed with the help of the L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ introduced in Proposition 1.4.

Recall the idea of the first half of the proof of Theorem 3.15 in Chapter 3: by Remark 2.10 in Chapter 3 we can find an isotopy of embeddings

$$\Sigma : NS \times [0,1] \to M$$

such that the restrictions to $NS \times \{0\}$, $NS \times \{1\}$ and $S \times \{t\}$ arbitrary are σ_0, σ_1 and id_S for arbitrary $t \in [0,1]$ respectively. Moreover an isotopy of embeddings

$$\Theta : W \times [0,1] \to NS$$

where W is an open neighbourhood of S in NS was constructed. The restriction of Θ to $W \times \{0\}$ is the identity on W and the restriction to $S \times \{t\}$ is id_S for arbitrary $t \in [0,1]$. Furthermore the composition of Θ with σ_0 is equal to the restriction of Σ to $W \times [0,1]$. We denote the restriction of Θ to $W \times \{t\}$ by Θ_t.

The manifold M is equipped with a Poisson bivector field Π. The embeddings σ_t give rise to a family of Poisson bivector field on NS which we denote by $\Pi_t := (\sigma_t)^*(\Pi|_{\sigma_t(NS)})$. The identity

$$\sigma_t|_W = (\sigma_0 \circ \theta_t)|_W$$

implies

$$\Pi_t|_W = (\theta_t)^* \left(\Pi_0|_{(\Theta_t)(W)}\right).$$

Using Lemma 6 in Chapter 6 we can find an open neigbourhood Y of S in NS and a smooth one-parameter family of embeddings

$$\Gamma : Y \times [0,1] \to NS$$

such that the restriction γ_t of Γ to $Y \times \{t\}$ is equal to $(\Theta_t|_{\gamma_t(Y)})^{-1}$. The relation

$$(\gamma_t)^*(\Pi_t|_{\gamma_t(Y)}) = \Pi_0|_Y$$

holds for all $t \in [0,1]$.

The connection ∇ on $\mathcal{E} \to E$ which was used to construct the L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ yields a connection on $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$. Using parallel transport with respect to this connection we obtain a smooth one-parameter family of isomorphisms $(\psi)_{t\in[0,1]}$ of bundles of graded algebras

$$\begin{tikzcd}
\wedge(\mathcal{E}|_Y \oplus \mathcal{E}|_Y^*) \arrow[r, "\psi_t"] \arrow[d] & \wedge(\mathcal{E}|_{\gamma_t(Y)} \oplus \mathcal{E}|_{\gamma_t(Y)}^*) \arrow[d] \\
Y \arrow[r, "\gamma_t"] & \gamma_t(Y).
\end{tikzcd}$$

This induces a one-parameter family of isomorphisms

$$\hat{\psi}_t : BFV(E)|_Y \xrightarrow{\cong} BFV(E)|_{\gamma_t(Y)}, \quad s \mapsto \psi_t \circ s \circ (\gamma_t^{-1}).$$

Define a graded Poisson bracket $[\cdot,\cdot](t)_{BFV}$ on $BFV(E)|_{\gamma_t(Y)}$ by

$$[\cdot,\cdot](t)_{BFV} := \hat{\psi}_t([\hat{\psi}_t^{-1}(\cdot), \hat{\psi}_t^{-1}(\cdot)]_{BFV}).$$

Suppose X and ζ are arbitrary sections of $\mathcal{E}|_{\gamma_t(Y)}$ and $\mathcal{E}^*|_{\gamma_t(Y)}$ arbitrary and compute

$$\begin{aligned}
[X,\zeta](t)_{BFV} + BFV_{\geq 1}(E) &= \hat{\psi}_t([\hat{\psi}_t^{-1}(X), \hat{\psi}_t^{-1}(\zeta)]_{BFV}) + BFV_{\geq 1}(E) \\
&= [\hat{\psi}_t^{-1}(X), \hat{\psi}_t^{-1}(\zeta)]_{BFV} \circ \gamma_t^{-1} + BFV_{\geq 1}(E) \\
&= [\hat{\psi}_t^{-1}(X), \hat{\psi}_t^{-1}(\zeta)]_{G} \circ \gamma_t^{-1} + BFV_{\geq 1}(E) \\
&= [X,\zeta]_G + BFV_{\geq 1}(E) \\
&= \zeta(X).
\end{aligned}$$

Here we used the fact that the induced connection on $\wedge(\mathcal{E} \otimes \mathcal{E}^*)$ is metric with respect to the fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$. This implies that the restriction of ψ_t to $\mathcal{E}^*|_Y$ is equal to the inverse of the adjoint of the restriction of ψ_t to $\mathcal{E}|_Y$.

Furthermore let f and g be two functions on $\gamma_t(Y)$ and calculate

$$\begin{aligned}
[f,g](t)_{BFV} + BFV_{\geq 1}(E) &= \hat{\psi}_t([\hat{\psi}_t^{-1}(f), \hat{\psi}_t^{-1}(g)]_{BFV}) + BFV_{\geq 1}(E) \\
&= \hat{\psi}_t([f \circ \gamma_t, g \circ \gamma_t]_{BFV}) + BFV_{\geq 1}(E) \\
&= \{f \circ \gamma_t, g \circ \gamma_t\}_{\Pi_0} \circ \gamma_t^{-1} \\
&= -[[\Pi_0, (\gamma_t)^*(f)]_{SN}], (\gamma_t)^*(g)]_{SN} \circ \gamma_t^{-1} \\
&= -[[(\gamma_t)^*(\Pi_t), (\gamma_t)^*(f)]_{SN}], (\gamma_t)^*(g)]_{SN} \circ \gamma_t^{-1} \\
&= -\left((\gamma_t)^*([[\Pi_t, f]_{SN}, g]_{SN}])\right) \circ \gamma_t^{-1} \\
&= -[[\Pi_t, f]_{SN}, g]_{SN} \\
&= \{f, g\}_{\Pi_t}.
\end{aligned}$$

Consequently $[\cdot,\cdot](t)_{BFV}$ is a restricted BFV-bracket for $(\gamma_t(Y), \Pi_t)$, in particular $[\cdot,\cdot](1)_{BFV}$ is a restricted BFV-bracket for $(\gamma_1(Y), \Pi_1)$. Moreover $\hat{\psi}_1(\Omega)$ is a Maurer–Cartan element of $(BFV(E)|_{\gamma_1(Y)}, [\cdot,\cdot](1)_{BFV})$. However it is not necessarily a restricted BFV-charge of $(BFV(E)|_{\gamma_1(Y)}, [\cdot,\cdot](1)_{BFV})$. Its component in ghost degree 1 is $\hat{\psi}_1(\Omega^0)$ which is not necessarily equal to Ω^0. The final step is to "gauge" $\hat{\psi}_1(\Omega)$ to a restricted BFV-charge of $(BFV(E)|_{\gamma_1(Y)}, [\cdot,\cdot](1)_{BFV})$.

By Lemma 6 in Chapter 6 there is an open neighbourhood Z of S in NS contained in all of $(\gamma_t(Y))_{t\in[0,1]}$. Remark 2.25 asserts that we can assume Z to be contractible along fibres. Consider the smooth one-parameter family of sections $(\hat{\psi}_t(\Omega^0))_{t\in[0,1]}$ over Z. Observe that the graph of $\Omega^0 \in \Gamma(\mathcal{E}|_Z)$ intersects the zero section $Z \hookrightarrow \mathcal{E}|_Z$ transversally, i.e.

$$T_x(\mathrm{graph}(\Omega^0)) + T_x Z = T_x\mathcal{E}$$

holds for all $x \in \mathrm{graph}(\Omega^0) \cap Z = S$. Diffeomorphisms map transversal intersections to transversal intersections and hence $\hat{\psi}_t(\Omega)$ intersects the zero section $Z \hookrightarrow \mathcal{E}|_Z$ transversally for all $t \in [0,1]$. Furthermore the intersection locus is still $S \hookrightarrow Z \hookrightarrow \mathcal{E}|_Z$. Hence we can apply Proposition 9 of Chapter 6 to $(\hat{\psi}_t(\Omega^0))$ and we obtain a smooth one-parameter family

$$(a_t)_{t\in[0,1]}$$

of sections of $\Gamma(\mathrm{End}(\mathcal{E}|_Z))$ such that the smooth one-parameter family $(A_t)_{t\in[0,1]}$ of sections of $\Gamma(GL_+(\mathcal{E}|_Z))$ satisfies

$$A_t \cdot \Omega^0 = \hat{\phi}_t(\Omega^0)$$

for arbitrary $t \in [0,1]$. Consider the smooth one-parameter family

$$(-A_t^{-1} \circ a_t \circ A_t)_{t\in[0,1]}$$

of sections of

$$\Gamma(\mathrm{End}(\mathcal{E}|_Z)) \cong \Gamma(\mathcal{E}|_Z \otimes \mathcal{E}^*|_Z).$$

It acts on $(BFV(E)|_Z, [\cdot,\cdot](1)_{BFV})$ via the adjoint action

$$[-A_t^{-1} \circ a_t \circ A_t, \cdot](1)_{BFV}.$$

Lemma 2.12 in Chapter 5 implies that this action integrates to a smooth one-parameter family of automorphisms α_t of the graded Poisson algebra

$$(BFV(E)|_Z, [\cdot,\cdot](1)_{BFV}).$$

Moreover for $X \in \Gamma(\mathcal{E}|_Z)$ arbitrary

$$[-A_t^{-1} \circ a_t \circ A_t, X](1)_{BFV} + BFV_{\geq 1}(E) = -(A_t^{-1} \circ a_t \circ A_t) \cdot X + BFV_{\geq 1}(E)$$

and consequently

$$\alpha(t)(\hat{\phi}_t(\Omega)) + BFV_{\geq 1}(E) = A_t^{-1} \cdot \hat{\phi}(\Omega) + BFV_{\geq 1}(E) = \Omega^0 + BFV_{\geq 1}(E)$$

holds for all $t \in [0,1]$. In particular

$$\Psi := \alpha_1 \circ \hat{\phi}_1$$

is an isomorphism of graded Poisson algebras that maps Ω to a restricted BFV-charge of $(BFV(E)|_Z, [\cdot,\cdot](1)_{BFV})$.

In summary we found two open neighbourhoods $U := \gamma_1^{-1}(Z)$ and $V := Z$ of S in NS and an isomorphism of graded algebras $\Psi : BFV(E)|_U \to BFV(E)|_V$ such that

$$\Psi \cdot (BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)|_U := (BFV_V(NS), \Phi([\Phi^{-1}(\cdot), \Phi^{-1}(\cdot)]_{BFV}), \Phi(\Omega|_U))$$

is a BFV-complex associated to (S, σ_1). $\square$

Definition 2.29. Let $(BFV(E), [\cdot,\cdot]_{BFV}, \Omega)$ be a BFV-complex associated to a coisotropic vector bundle (E, Π). The *germ* of $(BFV(E), [\cdot,\cdot]_{BFV}, \Omega)$ is the triple

$$(BFV^{\mathfrak{g}}(E), [\cdot,\cdot]^{\mathfrak{g}}_{BFV}, \Omega^{\mathfrak{g}})$$

where

(a) $BFV^{\mathfrak{g}}(E)$ is the graded algebra of equivalence classes of elements of $BFV(E)$ under the equivalence relation

$$F \sim G :\Leftrightarrow \text{there is an open neighbourhood } U \text{ of } S \text{ in } E \text{ such that } F|_U = G|_U,$$

(b) $[[F],[G]]^{\mathfrak{g}}_{BFV}$ is the equivalence class of $[F,G]_{BFV}$ in $BFV^{\mathfrak{g}}(E)$ and
(c) $\Omega^{\mathfrak{g}}$ is the equivalence class of Ω.

The *germ* of the differential graded Poisson algebra $(BFV(E), [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV})$ is the differential graded Poisson algebra

$$(BFV^{\mathfrak{g}}(E), [\Omega^{\mathfrak{g}},\cdot]_{BFV}, [\cdot,\cdot]^{\mathfrak{g}}_{BFV}).$$

REMARK 2.30. In Remark 2.27 we saw that the restriction of a BFV-complex $(BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)$ associated to (S,σ) in (M,Π) yields a morphism of differential graded Poisson algebras

$$r|_U : (BFV(NS), [\Omega,\cdot]_{SN}, [\cdot,\cdot]_{SN}) \to (BFV_U(NS), [\Omega|_U,\cdot]_{BFV}, [\cdot,\cdot]_{BFV}).$$

If U is assumed to be an open neighbourhood of S in NS which is contractible along fibres, then $r|_U$ is a quasi-isomorphism, i.e. it induces an isomorphism on cohomology.

The morphism $r|_U$ yields a morphism of graded Poisson algebras

$$r^{\mathfrak{g}} : (BFV(NS), [\Omega,\cdot]_{SN}, [\cdot,\cdot]_{SN}) \to (BFV^{\mathfrak{g}}(NS), [\Omega^{\mathfrak{g}},\cdot]^{\mathfrak{g}}_{BFV}, [\cdot,\cdot]^{\mathfrak{g}}_{BFV}).$$

In Remark 2.25 it was proved that the open neighbourhoods of S in NS that are contractible along fibres form a cofinal subset of the set of all open neighbourhoods of S in NS. This implies that $r^{\mathfrak{g}}$ induces an isomorphism on cohomology, i.e. $r^{\mathfrak{g}}$ is an quasi-isomorphism.

THEOREM 2.31. *Let S be a coisotropic submanifold of a Poisson manifold (M,Π) and $(BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)$ a BFV-complex associated to (S,σ) with σ an embedding of the normal bundle NS of S in M into M.*

Then the isomorphism class of the germ of $(BFV(NS), [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV})$ is an invariant of S, i.e. it does not depend on the specific choice of embedding σ, of the BFV-bracket $[\cdot,\cdot]_{BFV}$ and of the BFV-charge Ω.

PROOF. This is a Corollary of Theorem 2.28 and the definition of the germ of $(BFV(NS), [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV})$. $\square$

3. Relation to the homotopy Lie Algebroid

REMARK 3.1. Theorem 2.28 and Theorem 2.31 were presented in [**Sch2**]. Observe that although Theorem 2.31 is very similar to Theorem 3.15 in Chapter 3 it is slightly stronger: while the homotopy Lie algebroid yields an invariant that only depends on the fibre derivatives of the Poisson bivector field evaluated at the coisotropic submanifold, the BFV-complex produces an invariant that depends on the behaviour of the germ of the Poisson bivector field. The following example demonstrates this difference.

EXAMPLE 3.2. Consider the submanifold $S = \{0\}$ of $M = \mathbb{R}^2$ equipped with the Poisson bivector field

$$\Pi(x,y) := \begin{cases} 0 & (x,y) = (0,0) \\ \exp\left(-\frac{1}{x^2+y^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} & (x,y) \neq (0,0) \end{cases}.$$

This Poisson bivector field is symplectic on the complement of $\{0\}$. In particular 0 is the only point of $\mathbb{R}^2$ that is a coisotropic submanifold with respect to Π.

The fibre derivatives of Π evaluated at 0 are just the partial derivatives of Π at 0. Since Π vanished to infinite order at 0 all these derivatives vanish. Hence the

L_∞-algebra structure on $\Gamma(\wedge N_0\{0\}) \cong \wedge\mathbb{R}^2$ introduced in Corollary 3.4 in Chapter 3 is the trivial one.

On the other hand the restriction of Π to any open neighbourhood of $\{0\}$ in $\mathbb{R}^2$ is non-vanishing. Hence any BFV-bracket associated to any embedding of the normal bundle of $\{0\}$ in $\mathbb{R}^2$ into $\mathbb{R}^2$ is different from the trivial one, i.e. the one encoding the fibre pairing between $\mathbb{R}^2$ and $(\mathbb{R}^2)^*$.

Definition 3.3. A two-form $\tilde{\omega}$ on a manifold S is called *presymplectic* if $\tilde{\omega}$ is closed with respect to the de Rham differential and if the kernel of the vector bundle map

$$\tilde{\omega}^\# : TS \to T^*S$$

is a subvector bundle of TS.

Theorem 3.4. *Let S be a coisotropic submanifold of a symplectic manifold (M,ω) and denote the embedding $S \hookrightarrow M$ by ι. Then $\iota^*\omega$ is a presymplectic form on S.*

Given a manifold S equipped with a presymplectic form $\tilde{\omega}$ it is possible to find a symplectic manifold (M,ω) such that S is contained in M as a coisotropic submanifold and $\iota^*\omega$ is equal to $\tilde{\omega}$. Moreover any two such symplectic manifolds (M,ω) and (N,ω') are neighbourhood equivalent, i.e. one can find open neighbourhoods U and V of S in M and N respectively and a symplectomorphism

$$\psi : (U, \omega|_U) \xrightarrow{\cong} (V, \omega'|_V).$$

Remark 3.5. This Theorem due to Gotay [**Go**] implies that a behaviour such as in Example 3.2 does not appear in the realm of symplectic geometry since the symplectic form is determined by its restriction to the coisotropic submanifold up to neighbourhood equivalence. More precisely Gotay shows that for every coisotropic submanifold S of a symplectic manifold (M,ω) there is an embedding of the normal bundle NS of S in M such that the pull back of the symplectic structure to NS is polynomial in fibre directions, i.e. the symplectic structure on NS and hence on an open neighbourhood of S in M is determined by its fibre derivatives on S, see Remark 3.10 in Chapter 3.

The arguments in [**OP**] rely heavily on Theorem 3.4. Example 3.2 also demonstrates that this Theorem is not true for arbitrary Poisson manifolds.

Nevertheless the BFV-complex and the homotopy Lie algebroid are tightly related on an algebraic level:

Theorem 3.6. *Let (E,Π) be a coisotropic vector bundle and denote the zero section of E by S.*

The homotopy Lie algebroid

$$(\Gamma(\wedge E), (\lambda_n)_{n\geq 1})$$

associated to S – see Definition 3.6 in Chapter 3 – is L_∞ quasi-isomorphic to the differential graded Poisson algebra associated to a BFV-complex $(BFV(E), [\cdot,\cdot]_{BFV}, \Omega)$ of S, see Definition 2.15.

PROOF. The strategy of the proof is as follows: in the proof of Lemma 2.7 a homotopy $h : BFV(E) \to BFV(E)[-1]$ satisfying the relations

(i) $h \circ h = 0$,
(ii) $\wedge i^* \circ h = 0$,
(iii) $h \circ \wedge p^* = 0$ and
(iv) $\delta h + h \circ \delta = \mathrm{id} - (\wedge p^*) \circ (\wedge i^*)$

was constructed. Recall that

$$\wedge p^* : \Gamma(\wedge E) \to BFV(E)$$

denotes the fibrewise constant extension of sections of $\wedge E$ to sections of $\wedge \mathcal{E}$ followed by the inclusion into sections of $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$. The map

$$\wedge i^* : BFV(E) \to \Gamma(\wedge E)$$

is given by first restricting a section of $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$ to S followed by projection to a sections of $\wedge E$. Clearly $\wedge i^*$ and $\wedge p^*$ are chain maps. The identity $(\wedge i^*) \circ (\wedge p^*) = \mathrm{id}$ and property (iv) of the homotopy h imply that $\wedge i^*$ and $\wedge p^*$ induce inverse maps between $\Gamma(\wedge E)$ and $H(BFV(E), \delta = [\Omega^0, \cdot]_G)$. Hence

$$(\Gamma(\wedge E), 0) \underset{\wedge i^*}{\overset{\wedge p^*}{\rightleftarrows}} (BFV(E), \delta), h$$

provides contraction data, see Definition 2.1 in Chapter 2. According to Theorem 2.2 in Chapter 2 this contraction data can be used to perform homological transfer of L_∞-algebra structures along $\wedge i^*$. We apply this to the differential graded Poisson algebra

$$(BFV(E), [\Omega, \cdot]_{BFV}, [\cdot, \cdot]_{BFV})$$

associated to the BFV-complex $(BFV(E), [\cdot, \cdot]_{BFV}, \Omega)$. Denote the family of structure maps of the resulting L_∞-algebra structure on $\Gamma(\wedge E)$ by $(\gamma_n)_{n \geq 1}$. We claim that the structure maps $(\gamma_n)_{n \geq 1}$ are graded multiderivations of the graded algebra $\Gamma(\wedge E)$, i.e. they equip $\Gamma(\wedge E)$ with a P_∞-algebra structure. Consequently we can apply the arguments from the proof of Lemma 3.11 in Chapter 3: because $\Gamma(\wedge E)$ is locally generated by $\mathcal{C}^\infty(S)$ and $\Gamma(E)$ and the structure maps $(\gamma_n)_{n \geq 1}$ are multiderivations, it suffices to know the following values of the structure maps:

$$\gamma_n(\xi_1 \otimes \cdots \otimes \xi_n), \quad \gamma_n(\xi_1 \otimes \cdots \otimes \xi_{(n-1)} \otimes f), \quad \text{and} \quad \gamma_n(\xi_1 \otimes \cdots \xi_{(n-2)} \otimes f \otimes g)$$

where $\xi_1, \cdots, \xi_n$ are sections of E and f and g are functions on S. We will check by direct computation that these values of the structure maps coincide with the values found for the structure maps $(\lambda_n)_{n \geq 1}$ of the homotopy Lie algebroid of S, see Lemma 3.11 in Chapter 3.

First we aim to understand which decorated oriented trivalent trees contribute to the structure maps γ_n. Recall that the ghost-momentum degree is given by the decomposition

$$BFV(E) = \oplus_{q \geq 0} \Gamma(\wedge \mathcal{E} \otimes \wedge^q \mathcal{E}^*).$$

Furthermore we defined a family graded ideals

$$BFV_{\geq r}(E) := \Gamma(\wedge\mathcal{E} \otimes \wedge^{\geq r}\mathcal{E}^*).$$

Observe that $\wedge p^*(\Gamma(\wedge E)) \subset BFV(E)$ is concentrated in ghost-momentum degree 0. Moreover $\wedge i^*$ annihilates everything not of ghost-momentum degree 0. Consequently if a map associated to a tree with n leaves – i.e. exterior vertices with edges oriented away from them – maps tensor products of elements of $\wedge p^*(\Gamma(\wedge E))$ to $BFV_{\geq 1}(E)$, this tree does not contribute to γ_n. The map associated to a decorated oriented trivalent tree is given in terms of $[\cdot,\cdot]_{BFV}$, the "perturbation" of δ which is given by

$$D(\cdot) = [\Omega,\cdot]_{BFV} - \delta = [\Omega,\cdot]_{BFV} - [\Omega^0,\cdot]_G$$

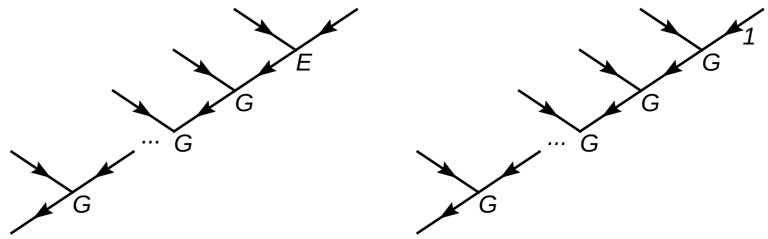
and the homotopy h. The homotopy h increases the ghost-momentum degree by 1, $D(\cdot)$ maps $BFV_{\geq r}(E)$ to $BFV_{\geq r}(E)$ and $[\cdot,\cdot]_{BFV}$ maps $BFV_{\geq r}(E) \times BFV_{\geq s}(E)$ to $BFV_{\geq (r+s-1)}(E)$ where the component of $[\cdot,\cdot]_{BFV}$ responsible for the shift of -1 is $[\cdot,\cdot]_G$. Now the number of copies of the homotopy decorating a given tree is equal to the number of copies of $[\cdot,\cdot]_{BFV}$ decorating that tree plus the number of copies of D decorating that tree minus one. Since the homotopy increases the ghost-momentum degree by 1 and $[\cdot,\cdot]_{BFV}$ is the only map that can decrease the ghost-momentum degree there must be at least as many copies of $[\cdot,\cdot]_{BFV}$ decorating a tree as there are copies of homotopies h decorating the tree. In Summary

$$\#([\cdot,\cdot]_{BFV}) \geq \#(h) = \#([\cdot,\cdot]_{BFV}) + \#(D) - 1$$

and so only trees with zero or one copy of D contribute to the structure maps $(\gamma_n)_{n\geq 1}$. Moreover it follows that in the case $\#(D) = 1$ only the part of $[\cdot,\cdot]_{BFV}$ that decreases the ghost-momentum degree contributes, i.e. instead of $[\cdot,\cdot]_{BFV}$ we can decorated these trees with $[\cdot,\cdot]_G$. In the case $\#(D) = 0$ one of the trivalent vertices decorated with $[\cdot,\cdot]_{BFV}$ might leave the ghost-momentum degree unchanged, i.e. we can decorate it with the component of $[\cdot,\cdot]_{BFV}$ of bidegree $(0,0)$ which we denote by $[\cdot,\cdot]_{\hat{\Pi}}$.

Summing up the previous paragraph one can adapt the construction of Remark 2.7 in Chapter 2 as follows: Consider oriented decorated trivalent trees whose edges are either all decorated by 0 or there is one exceptional edge decorated by 1. In the latter case all the trivalent vertices are decorated by $[\cdot,\cdot]_G$ and the exceptional edge is decorated by D. In case all edges are decorated by 0, one of the trivalent vertices is decorated by $[\cdot,\cdot]_{\hat{\Pi}}$ and all others are decorated by $[\cdot,\cdot]_G$. Then one places $-h$ between any two consecutive operations decorating the tree. In the end a copy of $\wedge p^*$ is placed at every leaf and $\wedge i^*$ is placed at the root, i.e. the unique exterior vertex with edge oriented towards it.

Observe that $\wedge p^*(\Gamma(\wedge E))$ is an abelian Lie subalgebra of $(BFV(E), [\cdot,\cdot]_G)$. This implies that the only two families of decorated oriented trivalent trees that contribute to $(\gamma_n)_{n\geq 1}$ are the following:



Here the decoration E refers to $[-,-]_{\hat{\Pi}}$, G refers to $[-,-]_G$ and the decoration of the edges is left out whenever it is zero. We denote the maps from $\Gamma(\wedge E)^{\otimes n}$ to $\Gamma(\wedge E)[2-n]$ corresponding to the tree on the left/right-hand side by L_n and R_n respectively. Up to skew-symmetrization and sign issues these two families of maps define the induced L_∞-algebra structure on $\Gamma(\wedge E)$.

Next we verify that $(L_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ are families of graded multiderivations. For ξ and ζ homogeneous elements of $\Gamma(\wedge E)$ we calculate

$$\begin{aligned}[\wedge p^*(\xi\wedge\zeta),\cdot]_G &= [(\wedge p^*\xi)\wedge(\wedge p^*\zeta),\cdot]_G\\ &= (\wedge p^*\xi)\wedge[\wedge p^*\zeta,\cdot]_G + (-1)^{|\xi||\zeta|}(\wedge p^*\zeta)\wedge[(\wedge p^*\xi),\cdot]_G\end{aligned}$$

and

$$h((\wedge p^*\xi)\wedge\cdot) = (-1)^{|\xi|}(\wedge p^*\xi)\wedge h(\cdot).$$

The last equality can be verified easily in local coordinates. In Lemma 10, Chapter 6 the signs for L_n and R_n are computed. For $\xi_1,\ldots,\xi_n$ homogeneous elements of $\Gamma(\wedge E)$ we obtain

$$\begin{aligned}&L_n(\xi_1\otimes\cdots\otimes\xi_n) =\\ &= (-1)^{l_n(\xi_1,\ldots,\xi_n)}\frac{1}{2}\wedge i^*\left([\wedge p^*\xi_1, h([\cdots h([\wedge p^*\xi_{(n-2)}, h([\wedge p^*\xi_{(n-1)},\wedge p^*\xi_n]_{\hat{\Pi}})]_G)\cdots]_G)]_G\right)\end{aligned}$$

with

$$l_n(\xi_1,\ldots,\xi_n) = n + \sum_{i=1}^{n-2}(n-i+1)(|\xi_1|+1)$$

and

$$\begin{aligned}&R_n(\xi_1\otimes\cdots\otimes\xi_n) =\\ &= (-1)^{r_n(\xi_1,\ldots,\xi_n)}\wedge i^*\left([\wedge p^*\xi_1, h([\cdots h([\wedge p^*\xi_{(n-1)}, hD(\wedge p^*\xi_n)]_G)\cdots]_G)]_G\right)\end{aligned}$$

with

$$r_n(\xi_1,\ldots,\xi_n) = (n-1) + \sum_{i=1}^{n-2}(n-i+1)(|\xi_1|+1).$$

The factor $\frac{1}{2}$ in front of the formula for L_n comes from the internal symmetry of the corresponding decorated oriented trivalent tree. Suppose $\xi_1,\ldots,\xi_{(n-1)}$ and ζ_1, ζ_2 are homogeneous sections of $\wedge E$. We want to compute

$$L_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes(\zeta_1\wedge\zeta_2)\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}) \quad\text{and}$$
$$R_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes(\zeta_1\wedge\zeta_2)\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}) \quad\text{respectively.}$$

Using the fact that $[\cdot,\cdot]_G$, $[\cdot,\cdot]_{\hat{\Pi}}$ and D are graded derivations, the identities

$$[\wedge p^*(\xi\wedge\zeta),\cdot]_G = (\wedge p^*\xi)\wedge[(\wedge p^*\zeta),\cdot]_G + (-1)^{|\xi||\zeta|}(\wedge p^*\zeta)\wedge[(\wedge p^*\xi),\cdot]_G,$$
$$h((\wedge p^*\xi)\wedge\cdot) = (-1)^{|\xi|}(\wedge p^*\xi)\wedge h(\cdot)$$

and that $\wedge p^*(\Gamma(\wedge E))$ is an abelian Lie subalgebra of $(BFV(E),[\cdot,\cdot]_G)$ yields

$$\begin{aligned}
&L_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes(\zeta_1\wedge\zeta_2)\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}) = \\
&= \ (-1)^{(|\xi_1|+\cdots+|\xi_{(i-1)}|+n)|\zeta_1|}\zeta_1\wedge L_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes\zeta_2\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}) \\
&\quad +(-1)^{(|\xi_1|+\cdots+|\xi_{(i-1)}|+|\zeta_1|+n)|\zeta_2|}\zeta_2\wedge L_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes\zeta_1\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)})
\end{aligned}$$

and

$$\begin{aligned}
&R_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes(\zeta_1\wedge\zeta_2)\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}) = \\
&= \ (-1)^{(|\xi_1|+\cdots+|\xi_{(i-1)}|+n)|\zeta_1|}\zeta_1\wedge R_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes\zeta_2\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}) \\
&\quad +(-1)^{(|\xi_1|+\cdots+|\xi_{(i-1)}|+|\zeta_1|+n)|\zeta_2|}\zeta_2\wedge R_n(\xi_1\otimes\cdots\otimes\xi_{(i-1)}\otimes\zeta_1\otimes\xi_i\otimes\cdots\otimes\xi_{(n-1)}).
\end{aligned}$$

These are exactly the signs in the definition of a graded skew-symmetric multi-derivation, see Remark 4.16 in Chapter 2.

Finally we compute the values of the structure maps $(L_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ on tensor products of smooth functions on S and sections on E respectively. Without loss of generality we can assume that $[\cdot,\cdot]_{\hat{\Pi}}$ is equal to $[\cdot,\cdot]_{i_\nabla(\Pi)}$ where $i_\nabla(\Pi)$ is the horizontal lift of the Poisson bivector field Π to a bivector field on $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$ with respect to a connection on this bundle induced from a connection ∇ on $E\to S$, see Proposition 1.8. Moreover we can assume without loss of generality that D is given by

$$[\Omega^0,\cdot]_{i_\nabla(\Pi)} - \frac{1}{2}[h([\Omega^0,\Omega^0]_{i_\nabla(\Pi)}),\cdot]_G,$$

see Lemma 2.19.

Let $(x^\beta)_{\beta=1,\dots,s}$ be local coordinates on S, $(y^i)_{i=1,\dots,e}$ linear coordinates along the fibres of E, $(b^i)_{i=1,\dots,e}$ the corresponding local frame on $\mathcal{E}^*$ and $(c_i)_{i=1,\dots,e}$ the dual frame on $\mathcal{E}$. These two frames yield a frame of $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$. We want to compute the values of L_n and R_n on tensor products of the following types

$$\begin{aligned}
A &:= i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_{(k-1)}})\otimes f\otimes i^*(c_{i_k})\otimes\cdots \\
&\qquad \cdots\otimes i^*(c_{i_{(k+l-1)}})\otimes g\otimes i^*(c_{i_{(k+l)}})\otimes\cdots\otimes i^*(c_{i_n}), \\
B &:= i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_{(k-1)}})\otimes f\otimes i^*(c_{i_k})\otimes\cdots\otimes i^*(c_{i_n}) \quad \text{and} \\
C &:= i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_n}) \quad \text{respectively.}
\end{aligned}$$

Here f and g are smooth functions on S. Observe that tensor products of type (A) are annihilated by R_n. For tensor products of type (B) and (C) both L_n and R_n may contribute. By Lemma 11 in Chapter 6 the identity

$$\wedge i^*\left([c_{i_1},h([\cdots h([c_{i_{k-1}},h([c_{i_k},h(X)]_G)]_G)\cdots]_G)]_G\right) = \frac{1}{k!}(i^*(c_{i_1})\cdots i^*(c_{i_k})\cdot X)|_S$$

holds for an arbitrary element X of $BFV(E)$ of ghost-momentum degree 0. We obtain

$$L_n(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-2)}}) \otimes f \otimes g) = (-1)^n \frac{1}{2(n-2)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-2)}} \cdot \{f,g\}_\Pi).$$

Graded skew-symmetrization of this term yields

$$\gamma_n(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-2)}}) \otimes f \otimes g) = (-1)^n \wedge i^*(c_{i_1} \cdots c_{i_{(n-2)}} \cdot (\{f,g\}_\Pi)).$$

Next we calculate

$$\begin{aligned} & L_n(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-1)}}) \otimes f) = \\ = \quad & (-1)^n \frac{1}{2(n-2)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-2)}} \cdot [c_{i_{(n-1)}}, f]_{i_\nabla(\Pi)}) \end{aligned}$$

and skew-symmetrization yields

$$\gamma_n^L(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-1)}}) \otimes f) = (-1)^n \left(\wedge i^* \sum_{k=1}^{n-1} (c_{i_1} \cdots \hat{c}_{i_k} \cdots c_{i_{(n-1)}} \cdot [c_{i_k}, f]_{i_\nabla(\Pi)})|_S \right).$$

Furthermore

$$\begin{aligned} & R_n(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-1)}}) \otimes f) = \\ = \quad & \frac{(-1)^{(n-1)}}{(n-1)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \cdot D(f)) \\ = \quad & \frac{(-1)^{(n-1)}}{(n-1)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \cdot [\Omega^0, f]_{i_\nabla(\Pi)}) \\ = \quad & \frac{(-1)^{(n-1)}}{(n-1)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \cdot \sum_{k=1}^{e} \left(y^k [c_k, f]_{i_\nabla(\Pi)} + c_k [y^k, f]_{i_\nabla(\Pi)} \right)) \\ = \quad & \frac{(-1)^{(n-1)}}{(n-1)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \cdot \sum_{k=1}^{e} \left(-y^k [c_k, f]_{i_\nabla(\Pi)} \right)) \\ & + \frac{(-1)^n}{(n-1)!} \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \Pi^\#(d_{DR} f)) \end{aligned}$$

and skew-symmetrization leads to

$$\begin{aligned} & \gamma_n^R(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-1)}}) \otimes f) = \\ = \quad & (-1)^{(n-1)} \wedge i^* \left(\sum_{k=1}^{n-1} c_{i_1} \cdots \hat{c}_{i_k} \cdots c_{i_{(n-1)}} \cdot [c_{i_k}, f]_{i_\nabla(\Pi)} \right) \\ & + (-1)^n \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \cdot \Pi^\#(d_{DR} f)) \end{aligned}$$

and consequently

$$\gamma_n(i^*(c_{i_1}) \otimes \cdots \otimes i^*(c_{i_{(n-1)}}) \otimes f) = (-1)^n \wedge i^*(c_{i_1} \cdots c_{i_{(n-1)}} \cdot \Pi^\#(d_{DR} f)).$$

Furthermore

$$L_n(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_n}))=\frac{(-1)^n}{2(n-2)!}\wedge i^*(c_{i_1}\cdots c_{i_{(n-2)}}\cdot[c_{i_{(n-1)}},c_{i_n}]_{i_\nabla(\Pi)})$$

and hence

$$\gamma_n^L(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_n}))=(-1)^n\wedge i^*\left(\sum_{k<l}c_{i_1}\cdots\hat{c}_{i_k}\cdots\hat{c}_{i_l}\cdots c_{i_n}\cdot[c_{i_k},c_{i_l}]_{i_\nabla(\Pi)}\right).$$

On the other hand

$$\begin{aligned}
R_n(c_{i_1}\otimes\cdots\otimes c_{i_n}) &= \frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(c_{i_1}\cdots c_{i_{(n-1)}}\cdot D(c_{i_n})\right)\\
&= \frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(c_{i_1}\cdots c_{i_{(n-1)}}\cdot([\Omega^0,c_{i_n}]_{i_\nabla(\Pi)}-\frac{1}{2}[h([\Omega^0,\Omega^0]_{i_\nabla(\Pi)}),c_{i_n}]_G)\right)\\
&= \frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(c_{i_1}\cdots c_{i_{(n-1)}}\cdot[\Omega^0,c_{i_n}]_{i_\nabla(\Pi)}\right)\\
&\quad+\frac{(-1)^n}{n!}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot(\frac{1}{2}[\Omega^0,\Omega^0]_{i_\nabla(\Pi)})\right)\\
&= \frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(c_{i_1}\cdots c_{i_{(n-1)}}\cdot\sum_{k=1}^{e}\left([y^kc_k,c_{i_n}]_{i_\nabla(\Pi)}\right)\right)\\
&\quad+\frac{(-1)^n}{n!}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\frac{1}{2}\sum_{k,l=1}^{e}\left([y^kc_k,y^lc_l]_{i_\nabla(\Pi)}\right)\right)\\
&= \frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(c_{i_1}\cdots c_{i_{(n-1)}}\cdot\sum_{k=1}^{e}y^k[c_k,c_{i_n}]_{i_\nabla(\Pi)}+c_k[y^k,c_{i_n}]_{i_\nabla(\Pi)}\right)\\
&\quad+\frac{(-1)^n}{n!}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\frac{1}{2}\sum_{k,l=1}^{e}y^ky^l[c_k,c_l]_{i_\nabla(\Pi)}\right)\\
&\quad+\frac{(-1)^n}{n!}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\frac{1}{2}\sum_{k,l=1}^{2}2y^k[c_k,y^l]_{i_\nabla(\Pi)}c_l+c_kc_l[y^k,y^l]_{i_\nabla(\Pi)}\right)\\
&= \frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(\sum_{r=1}^{(n-1)}c_{i_1}\cdots\hat{c}_{i_r}\cdots c_{i_{(n-1)}}\cdot[c_{i_r},c_{i_n}]_{i_\nabla(\Pi)}\right)\\
&\quad+\frac{(-1)^{(n-1)}}{(n-1)!}\wedge i^*\left(\sum_{k=1}^{e}c_{i_1}\cdots c_{i_{(n-1)}}\cdot c_k[y^k,c_{i_n}]_{i_\nabla(\Pi)}\right)\\
&\quad+\frac{(-1)^n}{n!}\wedge i^*\left(\sum_{r,s=1}^{n}c_{i_1}\cdots\hat{c}_{i_r}\cdots\hat{c}_{i_s}\cdots c_{i_n}\cdot[c_{i_r},c_{i_s}]_{i_\nabla(\Pi)}\right)
\end{aligned}$$

$$+\frac{(-1)^n}{n!}\wedge i^*\left(\sum_{r=1}^{n}\sum_{l=1}^{e} c_{i_1}\cdots\hat{c}_{i_r}\cdots c_{i_n}\cdot[c_{i_r},y^l]_{i_\nabla(\Pi)}c_l\right)$$
$$+\frac{(-1)^n}{n!}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\frac{1}{2}\sum_{k,l=1}^{e} c_k c_l[y^k,y^l]_{i_\nabla(\Pi)}\right)$$

and consequently

$$\begin{aligned}
&\gamma_n^R(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_n}))=\\
&= (-1)^{(n-1)}\wedge i^*\left(\sum_{r,s=1}^{(n-1)} c_{i_1}\cdots\hat{c}_{i_r}\cdots\hat{c}_{i_s}\cdots c_{i_n}\cdot[c_{i_r},c_{i_s}]_{i_\nabla(\Pi)}\right)\\
&\quad +(-1)^{(n-1)}\wedge i^*\left(\sum_{r=1}^{n}\sum_{k=1}^{e} c_{i_1}\cdots\hat{c}_{i_r}\cdots c_{i_n}\cdot c_k[y^k,c_{i_r}]_{i_\nabla(\Pi)}\right)\\
&\quad +(-1)^{n}\wedge i^*\left(\sum_{r<s}^{n} c_{i_1}\cdots\hat{c}_{i_r}\cdots\hat{c}_{i_s}\cdots c_{i_n}\cdot[c_{i_r},c_{i_s}]_{i_\nabla(\Pi)}\right)\\
&\quad +(-1)^{n}\wedge i^*\left(\sum_{r=1}^{n}\sum_{l=1}^{e} c_{i_1}\cdots\hat{c}_{i_r}\cdots c_{i_n}\cdot[c_{i_r},y^l]_{i_\nabla(\Pi)}c_l\right)\\
&\quad +(-1)^{n}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\Pi\right)\\
&= (-1)^{(n-1)}\wedge i^*\left(\sum_{r<s}^{(n-1)} c_{i_1}\cdots\hat{c}_{i_r}\cdots\hat{c}_{i_s}\cdots c_{i_n}\cdot[c_{i_r},c_{i_s}]_{i_\nabla(\Pi)}\right)\\
&\quad +(-1)^{n}\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\Pi\right).
\end{aligned}$$

All in all we obtain

$$\gamma_n^R(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_n}))=(-1)^n\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\Pi\right).$$

The three terms

$$\begin{aligned}
\gamma_n(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_{(n-2)}})\otimes f\otimes g) &= (-1)^n\wedge i^*(c_{i_1}\cdots c_{i_{(n-2)}}\cdot(\{f,g\}_\Pi)),\\
\gamma_n(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_{(n-1)}})\otimes f) &= (-1)^n\wedge i^*(c_{i_1}\cdots c_{i_{(n-1)}}\cdot\Pi^\#(d_{DR}f))\text{ and}\\
\gamma_n^R(i^*(c_{i_1})\otimes\cdots\otimes i^*(c_{i_n})) &= (-1)^n\wedge i^*\left(c_{i_1}\cdots c_{i_n}\cdot\Pi\right)
\end{aligned}$$

are exactly the coordinate expressions for the formulae of the structure maps $(\lambda_n)_{n\in\mathbb{N}}$ of the homotopy Lie algebroid found in Lemma 3.11 in Chapter 3.

We know from Section 1 in Chapter 2 that the induced L_∞-algebra on $\Gamma(\wedge E)$ which we identified with the structure maps of the homotopy Lie algebroid comes along with an L_∞-morphism to $(BFV(E),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV})$. We claim that this is in fact an L_∞ quasi-isomorphism, i.e. the chain map from $(\Gamma(\wedge E),\partial_\Pi)$ to $(BFV(E),[\Omega,\cdot]_{BFV})$ that is part of the L_∞-morphism induces an isomorphism on cohomology. This chain map is given by

$$\sum_{n\geq 0}(-h\circ D)^n\circ \wedge p^*.$$

The spectral sequence argument which we used to relate $H(BFV(E),[\Omega,\cdot]_{BFV})$ to $(\Gamma(\wedge E),\partial_\Pi)$ in the proof of Lemma 2.19 shows that every cocycle α of $(\Gamma(\wedge E),\partial_\Pi)$ extends to a cocycle $\hat{\alpha}$ of $(BFV(E),[\Omega,\cdot]_{BFV})$ such that $\wedge i^*(\hat{\alpha})=\alpha$. Moreover any cocycle of $(BFV(E),[\Omega,\cdot]_{BFV})$ can be obtained in this way and two different extensions of a cocycle in $(\Gamma(\wedge E),\partial_\Pi)$ are equal up to a coboundary in $(BFV(E),[\Omega,\cdot]_{BFV})$. This arguments imply that because $\sum_{n\geq 0}(-h\circ D)^n\circ\wedge p^*$ is a morphism of chain complexes satisfying $\wedge i^*\circ\left(\sum_{n\geq 0}(-h\circ D)^n\circ\wedge p^*\right)=\mathrm{id}$ it induces an isomorphism on cohomology. □

Theorem 3.7. *Given a coisotropic vector bundle (E,Π), the formal deformation problems associated to the L_∞-algebra $(\Gamma(\wedge E),(\lambda_n)_{n\geq 1})$ and to the differential graded Lie algebra $(BFV(E),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV})$ respectively are equivalent.*

Proof. This is a direct consequence of Theorem 3.6 and the general theory of formal deformation problems, see [**SS**] for instance. □

Corollary 3.8. *Let S be a coisotropic submanifold of a Poisson manifold (M,Π) and σ any embedding of the normal bundle NS to S in M into M whose restriction to S is the identity.*

The homotopy Lie algebroid $(\Gamma(\wedge NS),(\lambda_n^\sigma)_{n\geq 1})$ of (S,σ) in (M,Π) – see Definition 3.6 in Chapter 3 – is L_∞ quasi-isomorphic to the germ of the differential graded Poisson algebra

$$(BFV(NS),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV})$$

associated to a BFV-complex $(BFV(E),[\cdot,\cdot]_{BFV},\Omega)$ for (S,σ) introduced in Definition 2.18.

Proof. This is an immediate consequence of Theorem 3.6 and Remark 2.30. □

CHAPTER 5

Deformations

Following [**Sch3**] we investigate the geometric content of a BFV-complex associated to a coisotropic submanifold inside a Poisson manifold. In the first Section we introduce the set of coisotropic sections and normalized Maurer-Carten elements. It is proved that every coisotropic section can be lifted to a normalized Maurer–Cartan element, see Theorem 1.13. Moreover we present an example which was also considered in [**Z**] and [**OP**]. In Section 2 a natural equivalence relation on the set of coisotropic sections is introduced and the set of equivalence classes is computed for a Lagrangian submanifold of a symplectic manifold. We relax the notion of normalized Maurer–Cartan elements to geometric ones and construct a natural equivalence relation there. Theorem 2.25 asserts that the set of equivalence classes of geometric Maurer–Cartan elements is in bijection with the set of equivalence classes of coisotropic sections. This relation is extended to the level of groupoids in Section 3. The groupoids of coisotropic sections and of geometric Maurer–Cartan elements respectively are introduced. We construct a surjective morphism of groupoids from the latter groupoid to the former and determine its kernel. As a result we obtain an isomorphism of (quotient) groupoids, see Corollary 3.27.

1. Coisotropic Sections and (normalized) Maurer–Cartan elements

Remark 1.1. Let S be a coisotropic submanifold of a Poisson manifold (M,Π). In Section 2, Chapter 4 we established the existence of a BFV-complex

$$(BFV(NS), [\cdot,\cdot]_{BFV}, \Omega)$$

associated to (S,σ). Here σ is an embedding of the normal bundle NS of S into M such that the restriction $\sigma|_S$ is the identity on S – see in particular Definition 2.18, Definition 2.15 and Theorem 2.13 in Chapter 4. Moreover by Corollary 2.16 the isomorphism type of the differential graded Poisson algebra

$$(BFV(NS), [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV})$$

is an invariant of (S,σ). One of the aims of this and the following subsections is to investigate the geometric content of this differential graded Poisson algebra. We fix an embedding such as σ once and for all and work in the setting of coisotropic vector bundles introduced in Definition 2.9, Chapter 4.

Definition 1.2. Let (E,Π) be a coisotropic vector bundle. A section μ of $E \to S$ is *coisotropic* if its graph

$$S_\mu := \{(x,\mu(x)) : x \in S\}$$

is a coisotropic submanifold of (E, Π).

We denote the set of all coisotropic sections of (E, Π) by $\mathcal{C}(E, \Pi)$.

Definition 1.3. Let U be an open neighbourhood of the zero section S of a vector bundle $E \to S$ which is contractible along fibres, see Definition 2.24 in Chapter 4.

A section μ of E *lies in* U if for all $x \in S$ the value $\mu(x)$ of μ at x is an element of U.

Remark 1.4. Let L be a smooth manifold. The cotangent bundle T^*L carries a *canonical symplectic structure* given by

$$\omega_{\mathrm{can}} = -d_{DR}\theta_L$$

where θ_L is the *Liouville one-form*. It is defined as follows: consider the commutative diagram

$$\begin{tikzcd} & T(T^*L) \arrow[dl, "\pi_{T^*L}"'] \arrow[dr, "T(\pi_L)"] & \\ T^*L \arrow[dr, "\pi_L"'] & & TL \arrow[dl, "\pi_L"] \\ & L. & \end{tikzcd}$$

Now the Liouville one-form is given by

$$\theta_L : T(T^*L) \to \mathbb{R}, \quad X \mapsto < \pi_{T^*L}(X), T(\pi_L)(X) >$$

where $< \cdot, \cdot >$ denotes the fibre pairing between TL and T^*L. It has the property that, for any one-form μ on L, the pull back of θ_L along $\mu : L \to T^*L$ is μ again. The one-form θ_L can be universally characterized by this property, see [**Mi**] for instance. That $\omega_{\mathrm{can}} = -d_{DR}\theta_L$ is non-degenerate can be checked in local charts. The pull back property of θ_L implies that

$$\mu^*(\omega_{\mathrm{can}}) = -d_{DR}\mu$$

holds for all $\mu \in \Omega^1(L)$.

Theorem 1.5. *Let L be a Lagrangian submanifold of a symplectic manifold (M, ω), i.e. its dimension is half of the dimension of M and the pull back of the symplectic form to it vanishes. Then there is an open neighbourhood V of L in M and an open neighbourhood U of L in T^*L and a symplectomorphism*

$$\psi : (U, \omega|_\psi) \xrightarrow{\cong} (V, \omega_{\mathrm{can}}|_V).$$

Remark 1.6. Theorem 1.5 is an immediate the Darboux–Weinstein theorem. A proof can be found in [**W1**] for instance. It asserts that $L \hookrightarrow (T^*L, \omega_{\mathrm{can}})$ is a universal model for local properties of Lagrangian submanifolds inside symplectic manifolds.

EXAMPLE 1.7. Consider a Lagrangian submanifold L of a symplectic manifold (M, ω). Theorem 1.5 asserts that we can replace $L \hookrightarrow (M, \omega)$ by $L \hookrightarrow (T^*L, \omega_{\text{can}})$ as long as we are only interested in the symplectic geometry near L.

A section μ of T^*L is coisotropic inside $(T^*L, \omega_{\text{can}})$ if and only if its graph

$$L_\mu := \{(x, \mu(x)) : x \in L\}$$

is a coisotropic submanifold of $(T^*L, \omega_{\text{can}})$. Since the dimension of L_μ is equal to the dimension of L any such graph will automatically be a Lagrangian submanifold of $(T^*L, \omega_{\text{can}})$. Denote the inclusion of L_μ into T^*L by i_μ and the diffeomorphisms given by restricting

$$\mu : L \to T^*L$$

to its image L_μ by $\tilde{\mu}$.

A submanifold L_μ is Lagrangian if and only if

$$i_\mu^*(\omega_{\text{can}}) = 0.$$

Since $\tilde{\mu}$ is a diffeomorphism this is equivalent to

$$d_{DR}\mu = -\mu^*(\omega_{\text{can}}) = -\tilde{\mu}^*(i_\mu^* \omega_{\text{can}}) = 0,$$

i.e. a section μ is a coisotropic section of $(T^*L, \omega_{\text{can}})$ if and only if it is closed with respect to the de Rham differential.

REMARK 1.8. Oh and Park proved in [**OP**] that a generalization of the above picture holds for coisotropic submanifolds of symplectic manifolds: the de Rham complex $(\Gamma(\wedge T^*L), d_{DR})$ is replaced by the strong homotopy Lie algebroid

$$(\Gamma(\wedge NS), (\lambda_n^\sigma)_{n \in \mathbb{N}})$$

associated to (S, σ), see Corollary 3.4 and Definition 3.6 in Chapter 3. One can find an open neighbourhood U of S in NS such that a section μ that lies in U is a coisotropic section of (NS, Π_σ) if and only μ is a Maurer–Cartan element of the strong homotopy Lie algebroid, i.e. μ satisfies

$$\lambda_1(\mu) + \frac{1}{2}\lambda_2(\mu \otimes \mu) + \frac{1}{3!}\lambda_3(\mu \otimes \mu \otimes \mu) + \cdots = 0.$$

Instead of the Darboux–Weinstein Theorem one makes use of Gotay's Theorem (Theorem 3.4 in Chapter 4). Observe that one has to prove convergence of the above series in order to make sense of the Maurer–Cartan equation. An way to resolve this convergence issue is to consider formal solutions only, i.e. adjoin a formal variable ε and only allow for solutions in the ideal generated by ε.

In case S is a Lagrangian submanifold, one can find an embedding σ such that the strong homotopy Lie algebroid of (S, σ) is isomorphic to $(\Omega(L), d_{DR})$ and one recovers the characterization of coisotropic sections given in Example 1.7.

In the Poisson case the relation between Maurer–Cartan elements of the strong homotopy Lie algebroid $(\Gamma(\wedge NS), (\lambda_n^\sigma))$ and coisotropic sections that lie in an open neighbourhood of S in NS that is contractible along fibres does not hold.

Consider Example 3.2 in Chapter 4 where we presented a Poisson bivector field Π on $\mathbb{R}^2$ which is symplectic on $\mathbb{R}^2 \setminus \{0\}$ but vanishes at $\{0\}$. Hence 0 is the only point of $\mathbb{R}^2$ that is a coisotropic submanifold of $(\mathbb{R}^2, \Pi)$. Moreover we saw that the strong homotopy Lie algebroid yields the trivial L_∞-algebra structure on $\Gamma(\wedge N_0\{0\}) \cong \wedge\mathbb{R}^2$. In particular every element of $\mathbb{R}^2$ is a Maurer–Cartan element of this L_∞-algebra. Consequently every open neighbourhood of 0 in $\mathbb{R}^2$ contains an uncountable number of elements which satisfy the Maurer–Cartan equation but only $0 \in \mathbb{R}^2$ is coisotropic.

Definition 1.9. Let $(BFV(E), [\cdot,\cdot]_{BFV}, \Omega)$ be a BFV-complex associated to a coisotropic vector bundle (E, Π).

An element $\beta \in BFV^1(E)$ is an *algebraic Maurer–Cartan element* of (E, Π) if it satisfies the equation

$$[\Omega + \beta, \Omega + \beta]_{BFV} = 0.$$

We denote the set of all algebraic Maurer–Cartan element of (E, Π) by $\mathcal{D}_{\mathrm{alg}}(E, \Pi)$.

Remark 1.10. To be more precise we should associate the label "algebraic Maurer–Cartan element of" not to a coisotropic vector bundle (E, Π) but to a specific choice of BFV-complex for (E, Π). However by Corollary 2.16 in Chapter 4 different choices of BFV-complexes associated to (E, Π) lead to isomorphic sets of algebraic Maurer–Cartan elements. So let us fix a BFV-complex $(BFV(E), [\cdot,\cdot]_{BFV}, \Omega)$ associated to (E, Π) once and for all. Without loss of generality we may assume that the BFV-bracket $[\cdot,\cdot]_{BFV}$ is the one constructed with the help of the L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ introduced in Proposition 1.4 in Chapter 4.

The first algebraic Maurer–Cartan elements that come to one's mind are 0 and $-\Omega$. The former encodes the fact that the zero section S of E is a coisotropic submanifold of (E, Π) whereas the latter encodes the fact that E is a coisotropic submanifold of (E, Π). In general algebraic Maurer–Cartan elements can be interpreted as "coisotropic constraints" which do not necessarily yield submanifolds of E. Moreover the example $-\Omega$ shows that even if we restrict ourselves to algebraic Maurer–Cartan elements which come along with corresponding coisotropic *submanifolds*, the dimension of these submanifolds is not fixed.

We want to focus on a certain subset of $\mathcal{D}_{\mathrm{alg}}(E, \Pi)$ whose elements possess an interpretation in terms of coisotropic submanifolds, all of which are diffeomorphic. The key point is to introduce a normalization condition resembling the one we imposed on BFV-charges in part (ii) of Definition 2.12 in Chapter 4.

Definition 1.11. Let be a coisotropic vector bundle (E, Π).

An element $\beta \in \mathcal{D}_{\mathrm{alg}}(E)$ is a *normalized Maurer–Cartan element* of (E, Π) if its component in $BFV^{(1,0)}(E) = \Gamma(\mathcal{E})$ is equal to the pull back of some section of E.

We denote the set of all normalized Maurer–Cartan element of (E, Π) by $\mathcal{D}_{\mathrm{nor}}(E, \Pi)$.

REMARK 1.12. Recall that the bundle $\mathcal{E} \to E$ is the pull back of $E \to S$ along $E \xrightarrow{p} S$. Consequently there is a pull back map

$$p^* : \Gamma(E) \to \Gamma(\mathcal{E}).$$

The set $BFV^1(E)$ decomposes into

$$BFV^1(E) = \oplus_{k \geq 0} \Gamma(\wedge^{k+1}\mathcal{E} \otimes \wedge^k \mathcal{E}^*),$$

i.e. every element of $\mathcal{D}_{\mathrm{alg}}(E, \Pi) \subset BFV^1(E)$ has a component in $\Gamma(\mathcal{E})$.

THEOREM 1.13. *Let (E, Π) be a coisotropic vector bundle and μ a section of E. The following statements are equivalent:*

(a) *$-\mu$ is a coisotropic section of (E, Π).*
(b) *There is $\beta \in D_{\mathrm{nor}}(E, \Pi)$ whose component in $\Gamma(\mathcal{E})$ is equal to the pull back of μ.*

Furthermore given two elements β and β' of $D_{\mathrm{nor}}(E, \Pi)$ there is an automorphism Ψ of the graded Poisson algebra $(BFV(E), [\cdot, \cdot]_{BFV})$ that maps $\Omega + \beta$ to $\Omega + \beta'$.

PROOF. A direct verification can be found in [**Sch1**]. We will reduce the Theorem to Theorem 2.13 in Chapter 4. Let μ be a section of E. This induces a diffeomorphism φ_μ of E given by

$$\varphi_\mu : E \to E, \quad (x, e) \mapsto (x, e + \mu(x)).$$

Since φ_μ maps fibres of $E \to S$ to themselves, it induces an automorphism of the vector bundle $\mathcal{E}$: by definition $\mathcal{E}$ is the fibre product of E with itself over S, i.e.

$$\mathcal{E} := \{(e, f) \in E \times E : p(e) = p(f)\} \to E, \quad (e, f) \mapsto e.$$

The vector bundle structure is given by

$$\mathbb{R} \times \mathcal{E}_e \to \mathcal{E}_e, \quad (\lambda, (e, f)) \mapsto (e, \lambda \cdot f)$$

and

$$\mathcal{E}_e \times \mathcal{E}_e \to \mathcal{E}_e, \quad ((e, f), (e, g)) \mapsto (e, f + g).$$

Consider the map

$$\phi_\mu : \mathcal{E} \to \mathcal{E}, \quad (e, f) \mapsto (\varphi_\mu(e), f).$$

It is a vector bundle isomorphism of $\mathcal{E}$ covering φ_μ. Hence we obtain an induced automorphism of graded algebras

$$\hat{\phi}_\mu : BFV(E) \to BFV(E), \quad s \mapsto (\phi_\mu \circ s \circ \varphi_\mu^{-1}).$$

Because of

$$\begin{aligned} (\hat{\phi}_\mu(\Omega^0 + p^*(\mu)))(x, e) &= \phi_\mu((\Omega^0 + p^*(\mu))(x, e - \mu(x))) \\ &= \phi_\mu((x, e - \mu(x), e - \mu(x)) + (x, e - \mu(x), \mu(x))) \\ &= \phi_\mu(x, e - \mu(x), e) \\ &= (x, e, e) = (\Omega^0)(x, e) \end{aligned}$$

the identity

$$\hat{\phi}_\mu(\Omega^0 + p^*(\mu)) = \Omega^0$$

holds. Let $X \in \Gamma(\mathcal{E})$ and $\zeta \in \Gamma(\mathcal{E}^*)$ and compute

$$\begin{aligned} \hat{\phi}_\mu\left([\hat{\phi}_\mu^{-1}(X), \hat{\phi}_\mu^{-1}(\zeta)]_G\right) &= \left([\hat{\phi}_\mu^{-1}(X), \hat{\phi}_\mu^{-1}(\zeta)]_G\right) \circ \varphi_\mu^{-1} \\ &= [X, \zeta]_G = \zeta(X). \end{aligned}$$

Furthermore for arbitrary f and g in $\mathcal{C}^\infty(E)$ one obtains

$$\hat{\phi}_\mu\left(\{\hat{\phi}_\mu^{-1}(f), \hat{\phi}_\mu^{-1}(g)\}_\Pi\right) = (\varphi_\mu^{-1})^*\left(\{(\varphi_\mu)^*(f), (\varphi_\mu)^*(g)\}_\Pi\right).$$

This implies that

$$[\cdot,\cdot]^\mu_{BFV} := (\hat{\phi}_\mu)\left([\hat{\phi}_\mu^{-1}(\cdot), \hat{\phi}_\mu^{-1}(\cdot)]_{BFV}\right)$$

is a BFV-bracket for the coisotropic vector bundle (E, Π_μ) where Π_μ is the Poisson bivector field corresponding to the Poisson bracket on $\mathcal{C}^\infty(E)$ given by

$$\{f, g\}_{\Pi_\mu} := (\varphi_\mu^{-1})^*\left(\{(\varphi_\mu)^*(f), (\varphi_\mu)^*(g)\}_\Pi\right).$$

We denote the vanishing ideal of S in E by $\mathcal{I}_S$ and the vanishing ideal of

$$S_{-\mu} := \{(x, -\mu(x)) : x \in S\}$$

by $\mathcal{I}_{S_{-\mu}}$. Observe that

$$\varphi_\mu^*(\mathcal{I}_S) = \mathcal{I}_{S_{-\mu}}$$

holds. This implies

$$\{\mathcal{I}_S, \mathcal{I}_S\}_{\Pi_\mu} = (\varphi_\mu^{-1})^*\left(\{\mathcal{I}_{S_{-\mu}}, \mathcal{I}_{S_{-\mu}}\}_\Pi\right),$$

i.e. $\mathcal{I}_S$ is a coisotrope of $(\mathcal{C}^\infty(E), \{\cdot,\cdot\}_{\Pi_\mu})$ if and only if $\mathcal{I}_{S_{-\mu}}$ is a coisotrope of $(\mathcal{C}^\infty(E), \{\cdot,\cdot\}_\Pi)$. By Lemma 2.3 in Chapter 3 this implies that $S_{-\mu}$ is a coisotropic submanifold of (E, Π) if and only if S is a coisotropic submanifold of (E, Π_μ). By Theorem 2.13 in Chapter 4 the latter statement is equivalent to the existence of a BFV-charge for $(BFV(E), [\cdot,\cdot]^\mu_{BFV})$. Moreover all such BFV-charges are equivalent up to automorphisms of $(BFV(E), [\cdot,\cdot]^\mu_{BFV})$.

Let $\tilde{\Omega}$ be such a BFV-charge for $(BFV(E), [\cdot,\cdot]^\mu_{BFV})$. Consequently

$$\gamma := \hat{\phi}_\mu^{-1}(\tilde{\Omega}) - \Omega$$

is a Maurer–Cartan element of $(BFV(E), [\cdot,\cdot]_{BFV})$. Moreover we computed that

$$\hat{\phi}_\mu^{-1}(\Omega^0) = \Omega^0 + p^*(\mu)$$

holds, i.e. γ is a Maurer–Cartan element whose component in $\Gamma(\mathcal{E})$ is $p^*(\mu)$. Consequently $\hat{\phi}_\mu$ yields an isomorphism between BFV-charges of $(BFV(E), [\cdot,\cdot]^\mu_{BFV})$ and Maurer–Cartan elements of $(BFV(E), [\cdot,\cdot]_{BFV})$ whose component in $\Gamma(\mathcal{E})$ is equal to $p^*(\mu)$.

We conclude: $S_{-\mu}$ is a coisotropic submanifold of (E,Π) if and only if S is a coisotropic submanifold of (E,Π_μ). By Theorem 2.13 in Chapter 4, S is a coisotropic submanifold of (E,Π_μ) if and only if there is a BFV-charge for

$$(BFV(E),[\cdot,\cdot]^\mu_{BFV}).$$

Finally we established an isomorphism between the set of BFV-charges for the graded Poisson algebra $(BFV(E),[\cdot,\cdot]^\mu_{BFV})$ and the set of Maurer–Cartan elements of the BFV-complex $(BFV(E),[\cdot,\cdot]_{BFV})$ whose component in $\Gamma(\mathcal{E})$ is equal to $p^*(\mu)$. □

REMARK 1.14. The proof of Theorem 1.13 along with Theorem 2.13 and Lemma 2.11 in Chapter 4 can also be used to establish the following facts:

(a) The cohomology of $(BFV(E),[\Omega^0+p^*(\mu),\cdot]_G)$ is isomorphic to the graded algebra $\Gamma(\wedge E)$.
(b) Moreover a cocycle of $(BFV(E),[\Omega^0+p^*(\mu),\cdot]_G)$ is a coboundary if its component in ghost-momentum degree 0 vanishes when restricted to $S_{-\mu}$.
(c) $\Omega^0+p^*(\mu)$ extends to a Maurer–Cartan element of $(BFV(E),[\cdot,\cdot]_{BFV})$ if and only if

$$[\Omega^0+p^*(\mu),\Omega^0+p^*(\mu)]_{i_\nabla(\Pi)}$$

is exact with respect to $[\Omega^0+p^*,\cdot]_G$ or equivalently if and only if its restriction to $S_{-\mu}$ vanishes. The latter statement is true if and only if $S_{-\mu}$ is a coisotropic submanifold of (E,Π).

One can use these properties to construct Maurer–Cartan elements β of (E,Π) whose component in $\Gamma(\mathcal{E})$ is equal to $\Omega^0+p^*(\mu)$ in an iterative manner analogous to the construction of BFV-charges in the proof of Theorem 2.13 in Chapter 4.

Furthermore Theorem 1.13 yields a surjective map

$$L_{\mathrm{nor}}:\mathcal{D}_{\mathrm{nor}}(E,\Pi)\to\mathcal{C}(E,\Pi),\quad \beta\mapsto\beta^0=p^*(\mu)\mapsto-\mu.$$

Different elements in the preimage of some element of $\mathcal{C}(E,\Pi)$ under L_{nor} can be related by certain automorphisms of the differential graded Poisson algebra $(BFV(E),[\Omega,\cdot]_{BFV},[\cdot,\cdot]_{BFV})$. In the next Section a group will be constructed which acts on $\mathcal{D}_{\mathrm{nor}}(E,\Pi)$ and the orbits are exactly given by the preimages of elements of $\mathcal{C}(E,\Pi)$ under L_{nor}, i.e. L_{nor} induces an isomorphism between the quotient of $\mathcal{D}_{\mathrm{nor}}(E,\Pi)$ by the group and $\mathcal{C}(E,\Pi)$.

EXAMPLE 1.15. Consider the vector bundle

$$E:=(S^1)^4\times\mathbb{R}^2\to(S^1)^4$$

with coordinates $(\theta^1,\theta^2,\theta^3,\theta^4,x^1,x^2)$ where θ denotes the angle-coordinate on S^1. Equip E with the symplectic form

$$\omega=d_{DR}\theta^1\wedge d_{DR}x^1+d_{DR}\theta^2\wedge d_{DR}x^2+d_{DR}\theta^3\wedge d_{DR}\theta^4$$

and observe that (E,ω^{-1}) is a coisotropic vector bundle.

First let us construct a BFV-complex for (E, ω^{-1}). The pull back of $E \to (S^1)^4$ along $E \to (S^1)^4$ yields

$$E \times \mathbb{R}^2 \to E,$$

hence

$$BFV(E) = \Gamma(E \times \wedge(\mathbb{R}^2 \oplus (\mathbb{R}^*)^2) \to E).$$

We fix a frame (c_1, c_2) on $\mathbb{R}^2$ and denote the dual frame on $(\mathbb{R}^*)^2$ by (b^1, b^2). This induces a frame on $E \times (\wedge(\mathbb{R}^2 \oplus (\mathbb{R}^*)^2))$. Since the bundle $E \to (S^1)^4$ is trivial we can just set

$$[\cdot,\cdot]_{BFV} := [\cdot,\cdot]_G + \{\cdot,\cdot\}_{\omega^{-1}}.$$

Recall that $[\cdot,\cdot]_G$ is the graded Poisson bracket of degree 0 which encodes the fibre pairing between $\mathbb{R}^2$ and $(\mathbb{R}^*)^2$. The bracket $\{\cdot,\cdot\}_{\omega^{-1}}$ is the trivial lift of the Poisson bracket corresponding to ω^{-1} to a biderivation of $BFV(E)$. The tautological section Ω^0 of $E \times \mathbb{R}^2 \to E$ is given by

$$\Omega^0 = x^1 c_1 + x^2 c_2$$

and it is straightforward to check that

$$[\Omega^0, \Omega^0]_{BFV} = 0$$

holds, i.e. we may choose Ω^0 as a BFV-charge for $(BFV(E), [\cdot,\cdot]_{BFV})$. The differential $[\Omega^0, \cdot]_{BFV}$ reads

$$[\Omega^0, \cdot]_{BFV} = x^1 \frac{\partial}{\partial b^1} + x^2 \frac{\partial}{\partial b^2} + c_1 \frac{\partial}{\partial \theta^1} + c_2 \frac{\partial}{\partial \theta 2}$$

The second sheet of the spectral sequence introduced in the proof of Lemma 2.19 in Chapter 4 is

$$E_2 = \Gamma(\wedge(S^1)^4 \times (\wedge(\mathbb{R}^2)) \to (S^1)^4)$$

with differential $d_2 = c_1 \frac{\partial}{\partial \theta^1} + c_2 \frac{\partial}{\partial \theta^2}$. This differential can be identified with the de Rham differential d_{DR} on $\Omega(S^1 \times S^1)$ and consequently

$$H(BFV(E), [\Omega^0, \cdot]_{BFV}) \cong H(E_2, d_2) \cong H(S^1 \times S^1, \mathbb{R}) \otimes \mathcal{C}^\infty(S^1 \times S^1),$$

i.e. we obtain the tensor product of the Grassmann algebra generated by $[c_1]$ and $[c_2]$ and the algebra of smooth functions in the variables θ^3 and θ^4 with periodicity 2π.

Let $[\beta]$ be a cohomology class of degree $+1$ in $H(BFV(E), [\Omega^0, \cdot]_{BFV})$. The spectral sequence above implies that any such cohomology class can be represented by a cocycle $\beta \in BFV(E)$ whose component β^0 in $BFV^{(1,0)}(E) = \Gamma(E \times \mathbb{R}^2 \to E)$ is given by the pull back of a section of $(S^1)^4 \times \mathbb{R}^2 \to (S^1)^4$ which is closed with respect to d_2. Hence we may choose the cocycle $\beta \in BFV(E)$ such that

$$\beta^0 = f c_1 + g c_2$$

where f and g are smooth functions on $(S^1)^4$. Decompose β into components $\beta^0 + \beta^1$ with β^0 as above and $\beta^1 \in BFV^{(2,1)}(E)$, i.e. in coordinates we have

$$\beta^0 = \left(\sum_{k=1}^{2} \delta_k b^k\right) c_1 c_2$$

for a pair of smooth functions (δ_1, δ_2). Plugging this into $[\Omega^0, \beta]_{BFV} = 0$ yields

$$\left(\frac{\partial g}{\partial \theta^1} - \frac{\partial f}{\partial \theta^2}\right) c_1 c_2 + \left(\sum_{k=1}^{2} \delta_k x^k\right) c_1 c_2 = 0.$$

Consequently

$$\frac{\partial g}{\partial \theta^1} - \frac{\partial f}{\partial \theta^2} = -\sum_{k=1}^{2} \delta_k x^k$$

has to hold. Observe that the left-hand side is independent of (x^1, x^2), hence so is the right-hand side. In particular the right-hand sides is constant in (x^1, x^2) and we might evaluate it at $(x^1, x^2) = (0, 0)$, i.e. both sides of the equation must vanish independently. This implies that $(\delta_1, \delta_2) = (0, 0)$ and consequently $\beta^1 = 0$ and

$$\frac{\partial g}{\partial \theta^1} - \frac{\partial f}{\partial \theta^2} = 0.$$

What does the Maurer–Cartan equation for such a cocycle β amounts to? By definition

$$[\Omega^0 + \beta, \Omega^0 + \beta]_{BFV} = 0$$

and because β is closed with respect to $[\Omega^0, \cdot]_{BFV}$ this reduces to

$$[\beta, \beta]_{BFV} = 0.$$

Using the fact that $\beta = \beta^0$ we obtain

$$\{\beta^0, \beta^0\}_{\omega^{-1}} = 0.$$

Computing this expression yields

$$\{\beta^0, \beta^0\}_{\omega^{-1}} = 2\{f, g\}_{(S^1)^4}$$

with

$$\{f, g\}_{(S^1)^4} := \frac{\partial f}{\partial \theta^4}\frac{\partial g}{\partial \theta^3} - \frac{\partial g}{\partial \theta^3}\frac{\partial f}{\partial \theta^4}.$$

To sum up: a cocycle β whose component $\beta^0 = f c_1 + g c_2$ in $BFV^{(1,0)}(E)$ only depends on the angle-variables $(\theta^1, \theta^2, \theta^3, \theta^4)$ is a Maurer–Cartan element of the differential graded Lie algebra $(BFV(E), [\Omega^0, \cdot]_{BFV}, [\cdot, \cdot]_{BFV})$ if and only $\{f, g\}_{(S^1)^4}$ vanishes. This condition was also found in [**OP**] with the help of the homotopy Lie algebroid associated to (E, ω^{-1}).

When is a section μ of $E \to (S^1)^4$ a coisotropic section of (E, ω^{-1})? By Remark 1.14 it is coisotropic if and only if $\Omega^0 - p^*(\mu)$ can be extended to a Maurer–Cartan element of $(BFV(E), [\cdot,\cdot]_{BFV})$. Such an extension exists if and only if

$$[\Omega^0 - p^*(\mu), \Omega^0 - p^*(\mu)]_{\omega^{-1}}$$

is exact with respect to the differential $[\Omega^0 - p^*(\mu), \cdot]_G$. Let $p^*(\mu)$ be given by

$$p^*(\mu) = f c_1 + g c_2$$

where f and g are smooth functions on $(S^1)^4$ and consequently

$$\begin{aligned} R &:= [\Omega^0 - p^*(\mu), \Omega^0 - p^*(\mu)]_{\omega^{-1}} \\ &= 2c_1c_2 \left(\frac{\partial f}{\partial \theta^2} - \frac{\partial g}{\partial \theta^1} + \frac{\partial f}{\partial \theta^4}\frac{\partial g}{\partial \theta^3} - \frac{\partial g}{\partial \theta^4}\frac{\partial f}{\partial \theta^3} \right). \end{aligned}$$

The exactness condition translates into the existence of a pair of smooth functions (h_1, h_2) on E such that

$$R = [\Omega^0 - p^*(\mu), (h_1 b^1 + h_2 b^2)c_1c_2]_G = \big((x^1 - f)h_1 + (x^2 - g)h_2\big)\, c_1c_2$$

holds, i.e.

$$\frac{\partial f}{\partial \theta^2} - \frac{\partial g}{\partial \theta^1} + \frac{\partial f}{\partial \theta^4}\frac{\partial g}{\partial \theta^3} - \frac{\partial g}{\partial \theta^4}\frac{\partial f}{\partial \theta^3} = (x^1 - f)h_1 + (x^2 - g)h_2$$

has to hold for a pair of smooth functions (h_1, h_2) on E. The left-hand side of this equation is independent of the variables (x^1, x^2) and hence so is the right-hand side. Consequently we may evaluate the right-hand side at $(x^1, x^2) = (f, g)$ and so both sides of the equation have to vanish independently, i.e. a section $\mu = fc_1 + gc_2$ is coisotropic if and only if

$$\frac{\partial f}{\partial \theta^2} - \frac{\partial g}{\partial \theta^1} + \frac{\partial f}{\partial \theta^4}\frac{\partial g}{\partial \theta^3} - \frac{\partial g}{\partial \theta^4}\frac{\partial f}{\partial \theta^3} = 0$$

holds. This coincides with the condition found by Zambon by analytical considerations, see [**Z**].

Remark 1.16. Example 1.15 was first considered in [**Z**]. There it was used as a counterexample to show that the set of coisotropic sections does not form an infinite-dimensional manifold. In the Lagrangian case considered in Example 1.7 the space of coisotropic sections forms a linear subspace of $\Gamma(NL) \cong \Omega^1(L)$, i.e. the set of all Lagrangian submanifolds of a symplectic manifold (M, Ω) is locally modelled on vector spaces. Zambon used the above example to show that this does not hold for the set of coisotropic submanifolds: in Example 1.15 we proved that a section $\mu = fc_1 + gc_2$ is coisotropic if and only if

$$\frac{\partial f}{\partial \theta^2} - \frac{\partial g}{\partial \theta^1} + \frac{\partial f}{\partial \theta^4}\frac{\partial g}{\partial \theta^3} - \frac{\partial g}{\partial \theta^4}\frac{\partial f}{\partial \theta^3} = 0$$

holds. This is a nonlinear condition, hence given two coisotropic sections μ and ν, their sum fails in general to be a coisotropic section. Consequently the space of coisotropic submanifolds is not locally modelled on vector spaces.

In [**OP**] this phenomenon was given a conceptual explanation: as mentioned in Remark 1.8 the local model of the space of coisotropic submanifolds of a symplectic manifold near a fixed coisotropic submanifold S is given by the set of "small" Maurer–Cartan elements of the homotopy Lie algebroid $(\Gamma(\wedge NS), (\lambda_n^\sigma)_{n\geq 1})$ associated to (S, σ). This subset of $\Gamma(NS)$ is not a vector subspace, i.e. the set of all coisotropic submanifolds is not locally modelled on vector spaces.

2. Internal Symmetries and Moduli Spaces

Remark 2.1. In Definition 1.2 in the previous Section we introduced the notion of coisotropic sections of a coisotropic vector bundle (E, Π). In Section 1, Chapter 3 we saw that (E, Π) comes along with a group of inner symmetries $\mathrm{Ham}(E, \Pi)$, the group of Hamiltonian diffeomorphisms. By Lemma 2.9 in Chapter 3, $\mathrm{Ham}(E, \Pi)$ acts on the set of coisotropic submanifolds of (E, Π).

Definition 2.2. Let $E \to S$ be a vector bundle and K a smooth manifold (possible with boundary / corners). We denote the pull back of $E \to S$ along $E \times K \to E$ by E_K. A *smooth K-family of sections* of $E \to S$ is a section of E_K. In the special case $K = [0, 1]$ we also refer to sections of $E_{[0,1]}$ as *smooth one-parameter families of sections* of $E \to S$.

The restriction of a smooth K-family of sections $\hat{\mu}$ of $E \to S$ to $S \times \{k\} \cong S$ is denoted by μ_k and interpreted as a section of $E \to S$.

Definition 2.3. Given a coisotropic vector bundle (E, Π) a *Hamiltonian homotopy* of (E, Π) is a pair $(\hat{\mu}, \hat{\phi})$ where

(a) $\hat{\mu}$ is a smooth one-parameter family of sections of $E \to S$ whose restriction μ_t is a coisotropic section of (E, Π) for all $t \in [0, 1]$;
(b) $\hat{\phi}$ is a smooth one-parameter family of Hamiltonian diffeomorphisms of (E, Π), see Definition 1.15 in Chapter 3,

such that the graph S_{μ_t} of μ_t is equal to the image of S_{μ_0} under ϕ_t for arbitrary $t \in [0, 1]$.

Given a Hamiltonian homotopy $(\hat{\mu}, \hat{\phi})$ of (E, Π) we say that it is a Hamiltonian homotopy *from μ_0 to μ_1*.

Definition 2.4. Let (E, Π) be a coisotropic vector bundle. We define a relation $\sim_H$ on the set of coisotropic sections $\mathcal{C}(E, \Pi)$ of (E, Π) by

$$\mu \sim_H \nu :\Leftrightarrow \text{there is a Hamiltonian homotopy from } \mu \text{ to } \nu.$$

Lemma 2.5. *The relation $\sim_H$ on the set of coisotropic sections $\mathcal{C}(E, \Pi)$ of a coisotropic vector bundle (E, Π) is an equivalence relation.*

Proof. Reflexivity is obvious because any coisotropic sections μ comes along with a Hamiltonian homotopy

$$\mathrm{id}_\mu := ((\mu)_{t\in[0,1]}, (\mathrm{id})_{t\in[0,1]})$$

from μ to μ.

Given a Hamiltonian homotopy $(\hat{\mu}, \hat{\phi})$ from μ to ν, the pair

$$(\hat{\mu}, \hat{\phi})^{-1} := ((\mu_{(1-t)})_{t\in[0,1]}, (\phi_{(1-t)} \circ \phi_1^{-1})_{t\in[0,1]})$$

is a Hamiltonian homotopy from ν to μ.

Let $(\hat{\mu}, \hat{\phi})$ be a Hamiltonian homotopy from μ to ν and $(\hat{\nu}, \hat{\psi})$ a Hamiltonian homotopy from ν to λ. We want to construct a Hamiltonian homotopy from μ to λ. Choose a smooth function $\rho : [0,1] \to [0,1]$ with the following properties

(i) $\rho(0) = 0$ and $\rho(1) = 1$,
(ii) ρ is equal to $1/2$ on $[1/3, 2/3]$ and
(iii) the restriction of ρ to $]0, 1/3[$ and $]2/3, 1[$ are diffeomorphisms to $\rho(]0, 1/3[)$ and $\rho(]1/3, 1[)$ respectively.

The existence of such a function is demonstrated in Lemma 12 in Chapter 6. We define the pair

$$(\hat{\nu}, \hat{\psi})\Box_\rho(\hat{\mu}, \hat{\phi}) := (\hat{\nu}\Box_\rho\hat{\mu}, \hat{\psi}\Box_\rho\hat{\phi})$$

by

$$\begin{aligned}(\hat{\nu}\Box_\rho\hat{\mu})(t) \quad &:= \quad \begin{cases} \mu_{2\rho(t)} & 0 \le t \le 1/3 \\ \mu_1 & 1/3 \le t \le 2/3 \\ \nu_{(2\rho(t)-1)} & 2/3 \le t \le 1 \end{cases} \quad \text{and} \\ (\hat{\psi}\Box_\rho\hat{\phi})(t) \quad &:= \quad \begin{cases} \phi_{2\rho(t)} & 0 \le t \le 1/3 \\ \phi_1 & 1/3 \le t \le 2/3 \\ \psi_{(2\rho(t)-1)} \circ \phi_1 & 2/3 \le t \le 1 \end{cases} \quad \text{respectively.}\end{aligned}$$

Observe that $(\hat{\psi}\Box_\rho\hat{\phi})$ is a smooth one-parameter family of Hamiltonian diffeomorphisms generated by the smooth function $H : E \times [0,1] \to \mathbb{R}$ given by

$$H(x,t) := \begin{cases} 2\rho'(t)F(x, 2\rho(t)) & 0 \le t \le 1/3 \\ 0 & 1/3 \le t \le 2/3 \\ 2\rho'(t)G(x, 2\rho(t)-1), & 2/3 \le t \le 1 \end{cases}$$

where F and G denote the smooth functions whose Hamiltonian vector fields generate $\hat{\phi}$ and $\hat{\psi}$ respectively. It is straightforward to check that $(\hat{\nu}, \hat{\psi})\Box_\rho(\hat{\mu}, \hat{\phi})$ is a Hamiltonian homotopy from μ to λ. $\Box$

Definition 2.6. Let (E, Π) be a coisotropic vector bundle. The *moduli space of coisotropic sections* $\mathcal{M}(E, \Pi)$ of (E, Π) is the set of equivalence classes of elements of $\mathcal{C}(E, \Pi)$ with respect to the equivalence relation $\sim_H$.

Proposition 2.7. *Consider the coisotropic vector bundle $(T^*L, \omega_{\mathrm{can}}^{-1})$. The bijection*

$$\{\mu \in \Omega^1(L) : d_{DR}\mu = 0\} \xrightarrow{\cong} \mathcal{C}(T^*L \to L, \omega_{\mathrm{can}}^{-1})$$

established in Example 1.7 induces a bijection

$$H^1(L,\mathbb{R}) \xrightarrow{\cong} \mathcal{M}(T^*L \to L, \omega_{\mathrm{can}}^{-1}).$$

PROOF. Let μ and ν be two one-forms on L which are closed with respect to d_{DR}. We have to prove

$$[\mu] = [\nu] \in H^1(L,\mathbb{R}) \Leftrightarrow \text{there is a Hamiltonian homotopy from } \mu \text{ to } \nu.$$

$(\Rightarrow)$: Let A be a smooth function on L such that $d_{DR}A = \mu - \nu$. We claim that the Hamiltonian vector field $X_{\pi_L^*(A)}$ of the pull back of A along the projection $\pi_L : T^*L \to L$ is equal to

$$-p^*(d_{DR}A)$$

where we identify the pull back of $T^*L \to L$ along $\pi_L : T^*L \to L$ with the vertical part of the tangent bundle of T^*L. To verify this claim pick a local coordinate system $(q^i)_{i=1,\cdots,n}$ on L. This induces a local frame of TL, denote the dual frame on T^*L by $(p_i)_{i=1,\dots,n}$. The Liouville one-form θ_L is given by

$$\sum_{i=1}^n p_i d_{DR} q^i$$

and consequently ω_{can} reads

$$\sum_{i=1}^n d_{DR} q^i \wedge d_{DR} p_i.$$

The Hamiltonian vector field X_{p^*A} of $p^*(A)$ is

$$-[\omega^{-1}, p^*(A)]_{SN} = (\omega^{-1})^{\#}(d_{DR}(p^*A)) = (\omega^{\#})^{-1}(d_{DR}(p^*A))$$

and locally this amounts to

$$(\omega^{\#})^{-1}(\sum_{i=1}^n \frac{\partial A}{\partial q^i} d_{DR} q^i) = -\sum_{i=1}^n \frac{\partial A}{\partial q^i}\frac{\partial}{\partial p_i}.$$

The last term is the local expression of $-p^*(d_{DR}A)$ interpreted as a vertical vector field on T^*L.

Hence $X_{p^*(A)}$ generates the smooth one-parameter family of Hamiltonian diffeomorphisms

$$\varphi_t : T^*L \to T^*L, \quad (x,e) \mapsto (x, e - t(d_{DR}A)(x)).$$

This implies that

$$((\mu_t := \mu - t(d_{DR}A))_{t\in[0,1]}, (\varphi_t)_{t\in[0,1]})$$

is a Hamiltonian homotopy from μ to ν.

$(\Leftarrow)$: Suppose $(\hat{\mu}, \hat{\varphi})$ is a Hamiltonian homotopy from μ to ν. Let

$$F : M \times [0,1] \to \mathbb{R}$$

be the smooth function whose smooth one-parameter family of Hamiltonian vector fields generates $\hat{\varphi}$.

Because of

$$\begin{aligned}
\frac{d}{dt}|_{t=s}(\phi_t^*(\theta_L - p^*(\mu))) &= \phi_s^* \left(\mathcal{L}_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu)) \right) \\
&= \phi_s^* \left(d_{DR}(\iota_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu))) + \iota_{X_{F(\cdot,s)}} d_{DR}(\theta_L - p^*(\mu)) \right) \\
&= \phi_s^* \left(d_{DR}(\iota_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu))) - \iota_{X_{F(\cdot,s)}} \omega_{\text{can}} \right) \\
&= \phi_s^* \left(d_{DR}(\iota_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu))) - \omega_{\text{can}}^{\#}(X_{F(\cdot,s)}) \right) \\
&= \phi_s^* \left(d_{DR}(\iota_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu))) - d_{DR} F(\cdot, s) \right) \\
&= d_{DR} \left(\phi_s^*(\iota_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu)) - F(\cdot, s)) \right)
\end{aligned}$$

the identity

$$\begin{aligned}
\phi_1^*(\theta_L - p^*\mu) - (\theta_L - p^*(\mu)) &= \int_0^1 \frac{d}{dt}|_{t=s}(\phi_t^*\theta_L) ds \\
&= d_{DR} \left(\int_0^1 (\iota_{X_{F(\cdot,s)}}(\theta_L - p^*(\mu)) - F(\cdot, s)) \circ \phi_s ds \right) \\
&=: d_{DR} P
\end{aligned}$$

holds.

Consider the following smooth one-parameter of diffeomorphisms of L:

$$f_t : L \xrightarrow{\cong} L, \quad f_t := \pi_L \circ \phi_t \circ \mu.$$

It is straightforward to check that

$$\mu_t = \phi_t \circ \mu \circ f_t^{-1}$$

holds for all $t \in [0,1]$. We set

$$A : L \to \mathbb{R}, \quad A := P \circ \mu \circ f_1^{-1}$$

and calculate

$$\begin{aligned}
f_1^*(d_{DR} A) &= \mu^*(d_{DR} P) = (\phi_1 \circ \mu)^*(\theta_L - p^*(\mu)) - \mu^*(\theta_L - p^*(\mu)) \\
&= (\phi_1 \circ \mu)^*(\theta_L - p^*(\mu)) - \mu + \mu \\
&= (\phi_1 \circ \mu)^*(\theta_L - p^*(\mu)).
\end{aligned}$$

Consequently

$$\begin{aligned} d_{DR}A &= (\phi_1 \circ \mu \circ f_1^{-1})^*(\theta_L - p^*(\mu)) \\ &= (\nu)^*(\theta_L - p^*(\mu)) \\ &= \nu - \mu \end{aligned}$$

and hence $[\mu] = [\nu] \in H^1(L, \mathbb{R})$.

□

Remark 2.8. The second part of the proof of Proposition 2.7 essentially follows [**MS**]. Observe that it is crucial that $(T^*L, \omega_{\text{can}})$ is an exact symplectic manifold, i.e. the cohomology class of ω_{can} in $H^2(T^*L, \mathbb{R})$ vanishes.

By Theorem 1.5 Proposition 2.7 can be extended to the case of a Lagrangian submanifold L of arbitrary symplectic manifolds (M, ω) if one restricts attention to sections which are contained in a small open neighbourhood U of L in M which is contractible along fibres.

Observe that $H^1(L, \mathbb{R})$ is the moduli space of Maurer–Cartan elements of $(\Omega(L), d_{DR})$: The Maurer–Cartan elements of $(\Omega(L), d_{DR})$ are exactly the closed one-forms on L. Moreover $\Omega^0(L) = \mathcal{C}^\infty(L)$ acts on $\Omega^1(L)$ via

$$\Omega^0(L) \times \Omega^1(L) \to \Omega^1(L), \quad (f, \mu) \mapsto \mu + d_{DR}f.$$

This induces an action of $\Omega^0(L)$ on the set of closed one-forms on L. The quotient space of this action is $H^1(L, \mathbb{R})$.

This action of elements of degree 0 on the set of Maurer–Cartan elements generalizes to arbitrary L_∞-algebras. In particular the strong homotopy Lie algebroid $(\Gamma(\wedge E), (\lambda_n))$ associated to a coisotropic vector bundle (E, Π) and any BFV-complex $(BFV(E), [\cdot,\cdot]_{BFV}, \Omega)$ associated to (E, Π) come along with such an action on its set of Maurer–Cartan elements. In Section 1 we saw that the set of Maurer–Cartan elements of both structures contains information about the set of coisotropic sections $\mathcal{C}(E, \Pi)$ – for more precise statements we refer to Remark 1.8 and Theorem 1.13 in particular. It is natural to expect that the moduli space of coisotropic sections $\mathcal{M}(E, \Pi)$ is related to the quotient of the set of Maurer–Cartan elements by the action mentioned above.

Remark 2.9. Let (E, Π) be a coisotropic vector bundle and $[\cdot,\cdot]_{BFV}$ a BFV-bracket on $BFV(E)$. The set

$$\underline{BFV}(E) := \Gamma(\wedge(\mathcal{E}_{[0,1]} \oplus \mathcal{E}^*_{[0,1]}))$$

inherits the ghost degree, the ghost-momentum degree, the ghost-number, a filtration $\underline{BFV}_{\geq r}(E)$, the structure of a bigraded algebra and a graded Poisson algebra from $(BFV(E), [\cdot,\cdot]_{BFV})$, see Remark 2.3 in Chapter 4. Restriction of $\underline{BFV}(E)$ to $E \times \{t\}$ yields morphisms of graded Poisson algebras

$$\mathrm{ev}_t : (\underline{BFV}(E), [\cdot,\cdot]_{BFV}) \to (BFV(E), [\cdot,\cdot]_{BFV}).$$

DEFINITION 2.10. Let (E, Π) be a coisotropic vector bundle and $[\cdot,\cdot]_{BFV}$ a BFV-bracket on $BFV(E)$.

A *smooth one-parameter family of inner automorphisms* $\hat{\phi}$ of $(BFV(E), [\cdot,\cdot]_{BFV})$ is a morphism of graded Poisson algebras

$$\hat{\phi} : (BFV(E), [\cdot,\cdot]_{BFV}) \to (\underline{BFV}(E), [\cdot,\cdot]_{BFV})$$

satisfying

(a) the composition $\phi_0 := \mathrm{ev}_0 \circ \hat{\phi}$ is the identity of $BFV(E)$,
(b) the composition $\phi_t := \mathrm{ev}_t \circ \hat{\phi}$ is an automorphism of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ for $t \in [0,1]$ arbitrary and
(c) there is an element $\hat{\gamma} \in \underline{BFV}^0(E)$ such that for all $s \in [0,1]$ and all $\beta \in BFV(E)$
$$\frac{d}{dt}|_{t=s}(\phi_t(\beta)) = -([\gamma_s, \phi_s(\beta)]_{BFV})$$
holds. Here γ_s denotes the element $\mathrm{ev}_s(\hat{\gamma})$ of $BFV(E)$.

Denote the set of smooth one-parameter families of inner automorphism of the graded Poisson algebra $(BFV(E), [\cdot,\cdot])$ by $\underline{\mathrm{Inn}}(BFV(E))$.

An automorphism ϕ of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ is *inner* if there is a smooth one-parameter family of inner automorphisms $\hat{\phi}$ with $\phi_1 = \phi$. We denote the set of inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$ by $\mathrm{Inn}(BFV(E))$.

REMARK 2.11. Observe that this definition is totally analogous to the definition of smooth one-parameter families of Hamiltonian diffeomorphisms in Definition 1.15 in Chapter 3. The transition is given by associating the family of push forwards $((\phi_t)_* := (\phi_t^{-1})^*)_{t\in[0,1]}$ to a one-parameter family of Hamiltonian diffeomorphisms $(\phi_t)_{t\in[0,1]}$. Consequently Corollary 1.14 and Lemma 1.16 in Chapter 3 can be easily translated into statements about (smooth one-parameter families of) inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$.

In particular it turns out that Definition 2.10 is redundant: any smooth one-parameter family of automorphisms $\hat{\phi}$ of the *graded algebra* $BFV(E)$ generated by some $\hat{\gamma} \in \underline{BFV}^0(E)$ is automatically a smooth one-parameter family automorphisms of the *graded Poisson algebra* $(BFV(E), [\cdot,\cdot]_{BFV})$. Furthermore composition equips the sets $\underline{\mathrm{Inn}}(BFV(E))$ and $\mathrm{Inn}(BFV(E))$ with group structures.

The groups $\underline{\mathrm{Inn}}(BFV(E))$ and $\mathrm{Inn}(BFV(E))$ can be equipped with a filtration by subgroups

$$(\underline{\mathrm{Inn}}_{\geq r}(BFV(E)))_{r\geq 0} \quad \text{and} \quad (\mathrm{Inn}_{\geq r}(BFV(E)))_{r\geq 0}$$

respectively. $\underline{\mathrm{Inn}}_{\geq r}(BFV(E))$ is the group of smooth one-parameter families of inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$ generated by elements of $\underline{BFV}^0(E) \cap \underline{BFV}_{\geq r}(E)$. The group $\mathrm{Inn}_{\geq r}(BFV(E))$ is the image of $\underline{\mathrm{Inn}}_{\geq r}(BFV(E))$ under

$$\mathrm{ev}_1 : \underline{BFV}(E) \to BFV(E).$$

Observe that these two filtrations are bounded from below and above.

LEMMA 2.12. *Let (E,Π) be a coisotropic vector bundle. An element $\hat{\gamma} \in \underline{BFV}(E)$ integrates to a smooth one-parameter family of inner automorphisms of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ if and only if its component $\pi(\hat{\gamma})$ in*

$$\underline{BFV}^{(0,0)}(E) = \mathcal{C}^\infty(E \times [0,1])$$

integrates to a smooth one-parameter family of Hamiltonian diffeomorphisms of (E,Π).

PROOF. $(\Rightarrow)$: Suppose $\hat{\psi}$ is a smooth one-parameter family of inner automorphisms of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ generated by $\hat{\gamma}$. The decomposition

$$\underline{BFV}^0(E) = \oplus_{k\geq 0}\underline{BFV}^{(k,k)}(E)$$

yields a decomposition $\hat{\gamma} = \hat{\gamma}^0 + \hat{\gamma}^1 + \cdots$ and by definition $\hat{\gamma}^0 = \pi(\hat{\gamma})$. Because $\hat{\psi}$ is a morphism of algebras and preserves the total degree it maps the ideal I generated by

$$\Gamma(\wedge^{\geq 1}\mathcal{E} \otimes \wedge\mathcal{E}^*) + \Gamma(\wedge\mathcal{E} \otimes \wedge^{\geq 1}\mathcal{E}^*)$$

to the corresponding ideal $\underline{I}$ in $\underline{BFV}(E)$. Hence $\hat{\psi}$ factors through to a morphism of algebras

$$\hat{\zeta} : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E \times [0,1])$$

and since $\phi_t := \mathrm{ev}_t \circ \hat{\phi}$ is an automorphism of graded algebras, so is $\zeta_t := \mathrm{ev}_t \circ \hat{\zeta}$. Finally we calculate

$$\begin{aligned}
\frac{d}{dt}|_{t=s}\zeta_t(f) + \underline{I} &= -[\hat{\gamma}, \phi_t(f)]_{BFV} + \underline{I} = -[\hat{\gamma}^0, \phi_t(f)]_{BFV} + \underline{I} \\
&= -[\hat{\gamma}^0, \zeta_t(f)]_{i_\nabla(\Pi)} + \underline{I} = -\{\hat{\gamma}^0, \zeta_t(f)\}_\Pi + \underline{I}.
\end{aligned}$$

That implies that $\hat{\zeta}$ is generated by the adjoint action of $\hat{\gamma}^0$ and because of the Jacobi identity

$$\begin{aligned}
\frac{d}{dt}|_{t=s}\left(\zeta_t^{-1}(\{\zeta_t(\cdot), \zeta_t(\cdot)\}_\Pi)\right) &= (\zeta_s^{-1})\left(\{\gamma_s^0, \{\zeta_s(\cdot), \zeta_s(\cdot)\}_\Pi\}_\Pi\right) + \\
&-(\zeta_s^{-1})\left(\{\{\gamma_s^0, \zeta_s(\cdot)\}_\Pi, \zeta_s(\cdot)\}_\Pi\right) - (\zeta_s^{-1})\left(\{\zeta_s(\cdot), \{\gamma_s^0, \zeta_s(\cdot)\}_\Pi\}_\Pi\right)
\end{aligned}$$

vanishes, hence $(\zeta_t)_{t\in[0,1]}$ is a one-parameter family of automorphism of the Poisson algebra $(\mathcal{C}^\infty(E), \{\cdot,\cdot\}_\Pi)$. Lemma 1.10 in Chapter 3 asserts that there is a unique family of Poisson diffeomorphism $(\varphi_t)_{t\in[0,1]}$ such that $(\varphi_t^{-1})^* = \zeta_t$ for arbitrary $t \in [0,1]$. Moreover the family $(\varphi_t)_{t\in[0,1]}$ is a smooth one-parameter family of Hamiltonian diffeomorphisms generated by $\hat{\gamma}^0$.

($\Leftarrow$) : Assume that the smooth one-parameter family of Hamiltonian vector fields associated to $\hat{\gamma}^0 \in \mathcal{C}^\infty(E)$ integrates to a smooth one-parameter family of Hamiltonian diffeomorphisms $\hat{\varphi}$. In Remark 1.10 we fixed the BFV-bracket $[\cdot,\cdot]_{BFV}$ to be one constructed with the help of one of the L_∞ quasi-isomorphisms $\mathcal{L}_\nabla$ introduced in Proposition 1.4 in Chapter 4. Recall that ∇ is a connection on the vector bundle $E \to S$. It naturally extends to a connection on the vector bundle $\wedge(\mathcal{E}\oplus\mathcal{E}^*) \to E$. Using parallel transport with respect to this induced connection on $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$ yields a smooth one-parameter family of automorphisms of the vector bundle

$$\tilde{\varphi}_t : \wedge(\mathcal{E}\oplus\mathcal{E}^*) \to \wedge(\mathcal{E}\oplus\mathcal{E}^*)$$

covering $\varphi_t : E \to E$ and which also preserves the structure of $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$ as a bundle of bigraded algebras. In conclusion we obtain a morphism of bigraded algebras

$$\hat{\xi} : BFV(E) \to \underline{BFV}(E), \quad \beta \mapsto \tilde{\varphi}_t \circ \beta \circ \varphi_t^{-1}$$

such that $\xi_t := \mathrm{ev}_t \circ \hat{\xi}$ is an automorphism of bigraded algebras for all $t \in [0,1]$. The smooth one-parameter family $\hat{\xi}$ of automorphisms of $BFV(E)$ satisfies

$$\frac{d}{dt}|_{t=s}\xi_t(\cdot) = -\nabla_{X_{\gamma_s^0}}(\xi_s(\cdot))$$

where $\nabla_{X_{\gamma_s^0}}$ is the covariant derivative with respect to ∇ acting on $\Gamma(\wedge(\mathcal{E}\oplus\mathcal{E}^*))$. A proof of this fact can be found in [**Mi**] for instance.

Suppose $\hat{\phi}$ is a smooth one-parameter family of automorphisms of $BFV(E)$ starting at the identity and satisfying

$$\frac{d}{dt}|_{t=s}\phi_t = \Big(\xi_s^{-1}(-[\gamma_s,\cdot]_{BFV} + \nabla_{X_{\gamma_s^0}}) \circ \xi_s\Big) \circ \phi_s$$

for all $s \in [0,1]$. Setting $\psi_t := \xi_t \circ \phi_t$ yields a smooth one-parameter family of automorphisms starting at the identity and satisfying

$$\frac{d}{dt}|_{t=s}\psi_t(\cdot) = -[\gamma_s, \psi_s(\cdot)]_{BFV}$$

for all $s \in [0,1]$, i.e. $\hat{\gamma}$ integrates to a smooth one-parameter family of automorphisms of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ if and only if $\hat{\phi}$ exits.

We calculate

$$\begin{aligned} -[\gamma_s,\cdot]_{BFV} + \nabla_{X_{\gamma_s^0}} &= -[\gamma_s^0,\cdot]_{i_\nabla(\Pi)} - [\gamma_s^1,\cdot]_G + \nabla_{X_{\gamma_s^0}} + \cdots \\ &= -\nabla_{X_{\gamma_s^0}} - [\gamma_s^1,\cdot]_G + \nabla_{X_{\gamma_s^0}} + \cdots \\ &= -[\gamma_s^1,\cdot]_G + \cdots \end{aligned}$$

where $\cdots$ subsumes nilpotent derivations. This implies

$$\begin{aligned}\frac{d}{dt}|_{t=s}\phi_t &= \Big(\xi_s^{-1}(-[\gamma_s,\cdot]_{BFV}+\nabla_{X_{\gamma_s^0}}+\cdots)\circ\xi_s\Big)\circ\phi_s\\ &= \big(\xi_s^{-1}(-[\gamma_s^1,\cdot]_G+\cdots)\circ\xi_s\big)\circ\phi_s\\ &= \big(-[\xi_s^{-1}(\gamma_s^1),\cdot]_G+\cdots\big)\circ\phi_s\end{aligned}$$

where we used the fact that $\hat{\xi}$ is a smooth one-parameter family of automorphism of $(BFV(E),[\cdot,\cdot]_G)$ because the induced connection on $\wedge(\mathcal{E}\oplus\mathcal{E}^*)$ is metric with respect to the fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$.

Next we prove that there is a smooth one-parameter family $\hat{\chi}$ starting at the identity and satisfying

$$\frac{d}{dt}|_{t=s}\chi_t(\cdot) = -[\xi_s^{-1}(\gamma_s^1),\chi_s(\cdot)]_G$$

for all $s\in[0,1]$. Consider the action of $-[\xi_s^{-1}(\gamma_s)^1,\cdot]_G$ on $\Gamma(\mathcal{E}\oplus\mathcal{E}^*)$ which is given via

$$\begin{aligned}\mathcal{E}\otimes\mathcal{E}^*\to\mathrm{End}(\mathcal{E}) &\to \mathrm{End}(\mathcal{E})\oplus\mathrm{End}(\mathcal{E}^*)\to\mathrm{End}(\mathcal{E}\oplus\mathcal{E}^*)\\ \lambda &\mapsto \lambda-(\lambda)^*.\end{aligned}$$

This can be integrated fibrewise to a smooth one-parameter family of automorphisms of $\Gamma(\mathcal{E}\oplus\mathcal{E}^*)$. The natural extension of this smooth one-parameter family of automorphisms to a smooth-one parameter family of algebra automorphisms of $\Gamma(\wedge(\mathcal{E}\oplus\mathcal{E}^*))$ yields $\hat{\chi}$.

Finally observe that the integrability $\hat{\phi}$ can be reduced to the integrability of a nilpotent derivation with the help of $\hat{\chi}$ because

$$\begin{aligned}&\frac{d}{dt}|_{t=s}(\chi_t^{-1}\circ\phi_t)(\cdot) =\\ &= \chi_s^{-1}([\xi_s^{-1}(\gamma_s^1),\cdot]_G)\circ\phi_s-\chi_s^{-1}([\xi_s^{-1}(\gamma_s^1),\cdot]_G+\cdots)\circ\phi_s\\ &= \big(\chi_s^{-1}(\cdots)\chi_s\big)\circ(\chi_s^{-1}\circ\phi_s)=(\cdots)\circ(\chi_s^{-1}\circ\phi_s).\end{aligned}$$

Any smooth one-parameter family of nilpotent derivations can be integrated to a smooth one-parameter family of automorphisms. Consequently $\hat{\phi}$ exists and hence so does $\hat{\psi}$.

□

REMARK 2.13. Lemma 2.12 yields maps

$$\underline{L}:\underline{\mathrm{Inn}}(BFV(E))\to\underline{\mathrm{Ham}}(E,\Pi)\quad\text{and}\quad\underline{R}:\underline{\mathrm{Ham}}(E,\Pi)\to\underline{\mathrm{Inn}}(BFV(E)).$$

Here $\underline{L}$ is given by mapping the smooth one-parameter family of inner automorphisms of $(BFV(E),[\cdot,\cdot]_{BFV})$ integrating $\hat{\gamma}$ to the smooth one-parameter family of Hamiltonian diffeomorphisms integrating $\pi(\hat{\gamma})$ and $\underline{R}$ is given by mapping the smooth one-parameter family of Hamiltonian diffeomorphisms integrating $F\in\mathcal{C}^\infty(E\times[0,1])$ to the smooth one-parameter family of inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$ integrating $F \in \mathcal{C}^\infty(E \times [0,1]) = \underline{BFV}^{(0,0)}(E)$. Obviously $\underline{L} \circ \underline{R} = \mathrm{id}$, hence $\underline{L}$ is surjective and $\underline{R}$ is injective.

Lemma 2.14. *Given a coisotropic vector bundle (E, Π) the two maps*

$$\underline{L} : \underline{\mathrm{Inn}}(BFV(E)) \to \underline{\mathrm{Ham}}(E, \Pi), \quad \underline{R} : \underline{\mathrm{Ham}}(E, \Pi) \to \underline{\mathrm{Inn}}(BFV(E))$$

are morphisms of groups and the kernel of $\underline{L}$ is given by $\underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$.

Proof. Let $\hat{\varphi}$ and $\hat{\phi}$ be two smooth one-parameter families of Hamiltonian diffeomorphisms generated by the smooth functions F and G respectively. In Lemma 5 in Chapter 6 it was proved that $\hat{\phi} \circ \hat{\varphi}$ is the smooth one-parameter family of Hamiltonian diffeomorphisms generated by

$$H(x,t) := F(\phi_t^{-1}(x), t) + G(x,t).$$

Furthermore assume that $\hat{\Phi}$ and $\hat{\Psi}$ are two smooth one-parameter families if inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$ generated by $\hat{\gamma}$ and $\hat{\delta} \in \underline{BFV}^0(E)$ respectively. It is easy to check that

$$\frac{d}{dt}|_{t=s}(\hat{\Psi} \circ \hat{\Phi})(\cdot) = -[\Psi_s(\gamma_s) + \delta_s, (\hat{\Psi} \circ \hat{\Phi})(\cdot)]_{BFV}$$

holds for all $s \in [0,1]$. The component of $\Psi_s(\gamma_s) + \delta_s$ in $\underline{BFV}^{(0,0)}(E)$ is

$$(\psi_s^{-1})^*(\gamma_s^0) + \delta_s^0$$

where $\hat{\psi} := \underline{L}(\hat{\Psi})$.

These considerations immediately imply that $\underline{L}$ maps $\hat{\Psi} \circ \hat{\Phi}$ to $\underline{L}(\hat{\Psi}) \circ \underline{L}(\hat{\Phi})$ and $\underline{R}$ maps $\hat{\phi} \circ \hat{\varphi}$ to $\underline{R}(\hat{\phi}) \circ \underline{R}(\hat{\varphi})$.

It is obvious that $\underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$ lies in the kernel of $\underline{L}$ since every element of $\underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$ is generated by some $\hat{\gamma} \in \underline{BFV}_{\geq 1}(E)$ and consequently the component $\pi(\hat{\gamma})$ of $\hat{\gamma}$ in $\underline{BFV}^{(0,0)}(E)$ vanishes.

On the other hand suppose $\hat{\Phi}$ is a smooth one-parameter family of inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$ such that $\underline{L}(\hat{\Phi}) = (\mathrm{id}_E)_{t\in[0,1]}$ holds. Assume $\hat{\Phi}$ is generated by some $\hat{\gamma}$. Consequently the smooth one-parameter family of Hamiltonian vector fields associated to $\pi(\hat{\gamma})$ has to vanish. We decompose $\hat{\gamma}$ with respect to the ghost-momentum degree into

$$\hat{\gamma} = \hat{\gamma}^0 + \hat{\gamma}^1 + \hat{\gamma}^2 + \cdots .$$

We know that $X_{\gamma_s^0} = \Pi^\#(d_{DR}\gamma_s^0)$ vanishes for all $s \in [0,1]$. We claim that

$$[\hat{\gamma}^0, \cdot]_{BFV} = [\hat{\gamma}^0, \cdot]_G + [\hat{\gamma}^0, \cdot]_{i_\nabla(\Pi)} + \cdots$$

vanishes. Obviously the first term on the right-hand side of the equality vanishes. The second vanishes because

$$[\gamma_s^0, \cdot]_{i_\nabla(\Pi)} = i_\nabla(\Pi)(X_{\gamma_s^0}) = 0.$$

Denote the curvature of the connection on $\mathcal{E} \to E$ by R_∇ and pull it back to a two-form $\mathcal{R}_\nabla$ on $\mathcal{E}$ with values in $\mathrm{End}(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{E}^*$. Concerning the other contributions to $[\hat{\gamma}_0, \cdot]_{BFV}$ recall Remark 1.5 in Chapter 4: all the higher contributions to $[\cdot,\cdot]_{BFV}$ are given in terms of contractions of copies of $i_\nabla(\Pi)$ with copies of $\mathcal{R}_\nabla$. Hence γ_s^0 gets annihilated by any of this biderivations since all of the contributions are proportional to

$$< i_\nabla(\Pi), d_{DR}\gamma_s^0 >$$

which is equal to the pull back of $< \Pi, d_{DR}\gamma_s^0 >= 0$ to $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$.

Summing up: if the smooth-one parameter family of Hamiltonian vector fields associated to $\hat{\gamma}_0$ vanishes identically, the element $\hat{\gamma}_0$ is annihilated by $[\cdot,\cdot]_{BFV}$ and consequently

$$\begin{aligned} \frac{d}{dt}|_{t=s}\Phi_t(\cdot) &= -[\gamma_s, \Phi_s(\cdot)]_{BFV} \\ &= -[\gamma_s^0 + \gamma_s^1 + \gamma_s^2 + \cdots, \Phi_s(\cdot)]_{BFV} \\ &= -[\gamma_s^1 + \gamma_s^2 + \cdots, \Phi_s(\cdot)]_{BFV}, \end{aligned}$$

hence $\hat{\Phi}$ is generated by $\hat{\gamma}_1 + \hat{\gamma}_2 + \cdots$, i.e. $\hat{\Phi} \in \underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$. □

REMARK 2.15. Given a coisotropic vector bundle (E,Π) the group of automorphisms of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ acts on the set of algebraic Maurer–Cartan elements $\mathcal{D}_{\mathrm{alg}}(E,\Pi)$ introduced in Definition 1.9 by

$$\varphi \cdot \beta := \varphi(\Omega + \beta) - \Omega$$

where Ω is a fixed BFV-charge of $(BFV(E), [\cdot,\cdot]_{BFV})$. Consequently the group of inner automorphisms $\mathrm{Inn}(BFV(E))$ acts on $\mathcal{D}_{\mathrm{alg}}(E,\Pi)$ and so do all its subgroups $\mathrm{Inn}_{\geq r}(BFV(E))$.

PROPOSITION 2.16. *Let (E,Π) be a coisotropic vector bundle.*

The action of $\mathrm{Inn}_{\geq 2}(E,\Pi)$ on $\mathcal{D}_{\mathrm{alg}}(E,\Pi)$ restricts to an action on the set of normalized Maurer–Cartan elements $\mathcal{D}_{\mathrm{nor}}(E,\Pi)$ of (E,Π), see Definition 1.11.

Furthermore the map

$$L_{\mathrm{nor}} : \mathcal{D}_{\mathrm{nor}}(E,\Pi) \to \mathcal{C}(E,\Pi)$$

introduced in Remark 1.14 induces a bijection

$$[L_{\mathrm{nor}}] : \mathcal{D}_{\mathrm{nor}}(E,\Pi)/\,\mathrm{Inn}_{\geq 2}(E,\Pi) \xrightarrow{\cong} \mathcal{C}(E,\Pi).$$

PROOF. Suppose ϕ is an element of $\mathrm{Inn}_{\geq 2}(E,\Pi)$ and let $\hat{\phi}$ be an element of $\underline{\mathrm{Inn}}_{\geq 2}(BFV(E))$ with $\mathrm{ev}_1 \circ \hat{\phi} = \phi$. Let β be an element of $\mathcal{D}_{\mathrm{nor}}(E)$ with decomposition

$$\beta = \beta^0 + \beta^1 + \beta^2 + \cdots$$

where $\beta^k \in BFV^{(k+1,k)}(E)$. Because

$$\frac{d}{dt}|_{t=s}\phi_t(\Omega+\beta) = 0 + \underline{BFV}_{\geq 1}(E),$$

the component of $\hat{\phi}(\Omega+\beta)$ in $BFV^{(1,0)}(E)$ is constant and equal to $\Omega^0+\beta^0$. The normalization condition on elements of $\mathcal{D}_{\mathrm{nor}}(E,\Pi)$ is a condition on this component. Hence if the condition is satisfied for $t=0$ it will be satisfied for all $t \in [0,1]$, in particular it will be satisfied for $\phi(\Omega+\beta)$.

Since L_{nor} involves the projection $BFV^1(E) \to BFV^{(1,0)}(E)$, L_{nor} is constant along orbits of $\mathrm{Inn}_{\geq 2}(BFV(E))$. So it yields a map

$$[L_{\mathrm{nor}}] : \mathcal{D}_{\mathrm{nor}}(E,\Pi)/\,\mathrm{Inn}_{\geq 2}(BFV(E)) \to \mathcal{C}(E,\Pi)$$

which is surjective because L_{nor} is. In Theorem 1.13 it was proved that for any two normalized Maurer–Cartan elements β and β' whose images under L_{nor} coincide there is an automorphism ϕ of $(BFV(E),[\cdot,\cdot]_{BFV})$ that maps $\Omega+\beta$ to $\Omega+\beta'$, i.e.

$$\phi\cdot\beta = \beta'.$$

Moreover ϕ was constructed in such a way that it is manifestly an element of $\mathrm{Inn}_{\geq 2}(BFV(E))$. $\square$

Definition 2.17. Let (E,Π) be a coisotropic vector bundle. We define the set of *geometric Maurer–Cartan elements* of (E,Π) to be the orbit of the subset $\mathcal{D}_{\mathrm{nor}}(E,\Pi) \subset \mathcal{D}_{\mathrm{geo}}(E,\Pi)$, i.e.

$$\mathcal{D}_{\mathrm{geo}}(E,\Pi) := \mathrm{Inn}_{\geq 1}(BFV(E))\cdot\mathcal{D}_{\mathrm{nor}}(E,\Pi).$$

Lemma 2.18. *Given an algebraic Maurer–Cartan element β of a coisotropic vector bundle (E,Π) the following two statements are equivalent:*

(a) *β is a geometric Maurer–Cartan element.*
(b) *There is a section A of $GL_+(\mathcal{E})$ and a coisotropic section $-\mu$ of (E,Π) such that the component of β in $\Gamma(\mathcal{E})$ is equal to*

$$A(\Omega^0+p^*(\mu))-\Omega^0.$$

Moreover the coisotropic section $-\mu$ associated to a geometric Maurer–Cartan element β as above is unique.

Proof. First we prove the implication $(a) \Rightarrow (b)$. By definition there is $\phi \in \mathrm{Inn}_{\geq 1}(BFV(E))$ and $\beta' \in \mathcal{D}_{\mathrm{nor}}(E,\Pi)$ such that

$$\Omega+\beta = \phi(\Omega+\beta')$$

holds. Suppose $\hat{\phi}$ is an element of $\underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$ such that $\mathrm{ev}_1\circ\hat{\phi}=\phi$ and let $\hat{\gamma}$ be the element of $\underline{BFV}^0(E)$ that generates $\hat{\phi}$. We calculate

$$\begin{aligned}
\frac{d}{dt}|_{t=s}(\phi_t)(\Omega+\beta')+\underline{BFV}_{\geq 1}(E) &= -[\gamma_s,(\phi_t)(\Omega+\beta')]_{BFV}+\underline{BFV}_{\geq 1}(E)\\
&= -[\gamma_s^1,(\phi_t)(\Omega^0+\beta'^0)]_G+\underline{BFV}_{\geq 1}(E).
\end{aligned}$$

Hence the induced action of $\hat{\phi}$ on $\Gamma(\mathcal{E})$ is given by the smooth one-parameter family of fibrewise linear automorphisms $\hat{A} \in \Gamma(GL_+(\mathcal{E})_{[0,1]})$ of $\mathcal{E}$ integrating the smooth one-parameter family

$$-\gamma_s^1 \in \underline{BFV}^{(1,1)}(E) = \Gamma(\mathcal{E}_{[0,1]} \otimes \mathcal{E}^*_{[0,1]}) \cong \Gamma(\mathrm{End}(\mathcal{E})_{[0,1]}).$$

Consequently

$$\Omega^0 + \beta^0 = A_1(\Omega^0 + \beta'^0).$$

Since $\beta' \in \mathcal{D}_{\mathrm{nor}}(E, \Pi)$ there is a coisotropic section $-\mu$ such that $\beta' = p^*(\mu)$, see Theorem 1.13.

Next we prove $(b) \Rightarrow (a)$: Given $-\mu$ a coisotropic section of (E, Π) and $A \in \Gamma(GL_+(\mathcal{E}))$ such that

$$\Omega^0 + \beta^0 = A(\Omega^0 + p^*(\mu))$$

holds. By Theorem 1.13 there is a Maurer–Cartan element β' of (E, Π) with $\beta'^0 = p^*(\mu)$. Let $\hat{A}$ be a smooth one-parameter family of sections of $GL_+(\mathcal{E})$ connecting the identity with A. Setting

$$a_s := \left(\frac{d}{dt}|_s A_t\right) \circ (A_s)^{-1}$$

yields a smooth one-parameter family $\hat{a}$ of sections of $\mathrm{End}(\mathcal{E})$ that integrates to $\hat{A}$. Let $\hat{\alpha}$ be the smooth one-parameter family of inner automorphisms of $(BFV(E), [\cdot,\cdot]_{BFV})$ generated by $\hat{a}$. Lemma 2.12 assures that this smooth one-parameter family $\hat{\alpha}$ exists. We calculate

$$(\alpha_1)^{-1}(\Omega + \beta) + BFV_{\geq 1}(E) = (a_1)^{-1}(\Omega^0 + \beta^0) = \Omega^0 + p^*(\mu),$$

i.e. the Maurer–Cartan elements $(\alpha_1)^{-1} \cdot \beta$ and β' are both normalized and have the same image under L_{nor}. By Theorem 2.16 there is $\phi \in \mathrm{Inn}_{\geq 1}(BFV(E))$ with $\phi \cdot \beta' = (\alpha_1)^{-1} \cdot \beta$. Consequently

$$\beta = (\alpha \circ \phi) \cdot \beta'$$

holds. Observe that $\beta' \in \mathcal{D}_{\mathrm{nor}}(E, \Pi)$ and α, $\beta \in \mathrm{Inn}_{\geq 1}(BFV(E))$ imply $\beta \in \mathcal{D}_{\mathrm{geo}}(E, \Pi)$.

The uniqueness of the coisotropic section $-\mu$ associated to a geometric Maurer–Cartan element β is established as follows: The vanishing locus of $A(\Omega^0 + p^*(\mu))$ for $A \in \Gamma(GL_+(\mathcal{E}))$ is given by

$$S_{-\mu} := \{(x, -\mu(x)) : x \in S\}.$$

Given any geometric Maurer–Cartan element β the vanishing locus of $\Omega^0 + \beta^0$ is equal to $S_{-\mu}$ for some coisotropic section $-\mu$ of (E, Π). Since a section can be reconstructed from its graph $-\mu$ is uniquely determined by β. □

COROLLARY 2.19. *Given a coisotropic vector bundle* (E,Π) *the mapping*

$$L_{\mathrm{nor}} : \mathcal{D}_{\mathrm{nor}}(E,\Pi) \to \mathcal{C}(E,\Pi)$$

introduced in Remark 1.14 uniquely extends in an $\mathrm{Inn}_{\geq 1}(BFV(E))$*-invariant way to a map*

$$L_{\mathrm{geo}} : \mathcal{D}_{\mathrm{geo}}(E,\Pi) \to \mathcal{C}(E,\Pi)$$

which induces a bijection

$$[L_{\mathrm{geo}}] : \mathcal{D}_{\mathrm{geo}}(E,\Pi)/\,\mathrm{Inn}_{\geq 1}(BFV(E)) \xrightarrow{\cong} \mathcal{D}(E,\Pi).$$

PROOF. Because of $\mathcal{D}_{\mathrm{geo}}(E,\Pi) = \mathrm{Inn}_{\geq 1}(BFV(E))\cdot\mathcal{D}_{\mathrm{nor}}(E,\Pi)$ there is at most one $\mathrm{Inn}_{\geq 1}(BFV(E))$-invariant way to extend L_{nor} to $\mathcal{D}_{\mathrm{geo}}(E,\Pi)$. One way to write L_{geo} down is

$$\beta \mapsto (\text{vanishing set of } \Omega^0 + \beta^0) = S_{-\mu} \mapsto -\mu,$$

see also the part of the proof of Lemma 2.18 concerning the uniqueness of $-\mu$ associated to β. It is straightforward to check that this map is $\mathrm{Inn}_{\geq 1}(BFV(E))$-invariant.

Surjectivity of $[L_{\mathrm{geo}}]$ is evident. Now suppose β and β' are two geometric Maurer–Cartan elements of (E,Π) that get mapped to the same coisotropic section $-\mu$ under L_{geo}. By definition there are two inner automorphisms ϕ, $\phi' \in \mathrm{Inn}_{\geq 1}(BFV(E))$ such that $\phi\cdot\beta$ and $\phi'\cdot\beta'$ are normalized Maurer–Cartan elements. Since L_{geo} is invariant under the action by $\mathrm{Inn}_{\geq 1}(BFV(E))$ on $\mathcal{D}_{\mathrm{geo}}(E,\Pi)$ the two normalized Maurer–Cartan elements $\phi\cdot\beta$ and $\phi'\cdot\beta'$ both get mapped to $-\mu$ by L_{nor}. Lemma 2.16 implies that there is an inner automorphism $\varphi \in \mathrm{Inn}_{\geq 2}(BFV(E))$ such that $\varphi\cdot(\phi\cdot\beta) = \phi'\cdot\beta'$, i.e.

$$\beta' = (\phi'^{-1}\circ\varphi\circ\phi)\cdot\beta.$$

Observe that $\phi'^{-1}\circ\varphi\circ\phi$ lies in $\mathrm{Inn}_{\geq 1}(BFV(E))$. $\square$

DEFINITION 2.20. Given a coisotropic vector bundle (E,Π), a *gauge homotopy* is a pair $(\hat{\beta},\hat{\phi})$ where

(a) $\hat{\beta}$ is a smooth one-parameter family of sections of $\wedge(\mathcal{E}\oplus\mathcal{E}^*) \to E$ whose restriction β_t is a geometric Maurer–Cartan element for all $t \in [0,1]$;
(b) $\hat{\phi}$ is a smooth one-parameter family of inner automorphisms of the differential graded Poisson algebra $(BFV(E),[\cdot,\cdot]_{BFV})$, see Definition 2.10,

such that $\phi_t\cdot\beta^0 = \beta^t$ holds for arbitrary $t \in [0,1]$.

Given a gauge homotopy $(\hat{\beta},\hat{\phi})$ of (E,Π) we say that it is a gauge homotopy from β^0 to β^1.

DEFINITION 2.21. Let (E,Π) be a coisotropic vector bundle. We define a relation $\sim_G$ on the set of geometric Maurer–Cartan elements $\mathcal{D}_{\mathrm{geo}}(E,\Pi)$ of (E,Π) via

$$\beta \sim_G \gamma :\Leftrightarrow \text{there is a gauge homotopy from } \beta \text{ to } \gamma.$$

LEMMA 2.22. *The relation $\sim_G$ on the set of geometric Maurer–Cartan elements $\mathcal{D}_{\mathrm{geo}}(E,\Pi)$ of a coisotropic vector bundle (E,Π) is an equivalence relation.*

PROOF. The proof can be copied mutatis mutandis from the proof of Lemma 2.5. In particular

(a) to every geometric Maurer–Cartan element β we associate

$$\mathrm{id}_\beta := ((\beta)_{t\in[0,1]}, (\mathrm{id})_{t\in[0,1]})$$

which is a gauge homotopy from β to β,

(b) given a gauge homotopy $(\hat{\beta},\hat{\phi})$ from β to γ

$$(\hat{\beta},\hat{\phi})^{-1} := ((\beta_{(1-t)})_{t\in[0,1]}, (\phi_{(1-t)}\circ\phi_1^{-1})_{t\in[0,1]})$$

is a gauge homotopy from γ to β and

(c) any choice of a smooth function $\rho:[0,1]\to[0,1]$ satisfying
  (i) $\rho(0)=0$ and $\rho(1)=1$,
  (ii) ρ is equal $1/2$ on $[1/3,2/3]$ and
  (iii) the restriction of ρ to $]0,1/3[$ and $]2/3,1[$ are diffeomorphisms to $\rho(]0,1/3[)$ and $\rho(]1/3,1[)$ respectively.

allows us to define an operation $\Box_\rho$ on the space of "composable" gauge homotopies, i.e. given $(\hat{\alpha},\hat{\varphi})$ a gauge homotopy from α to β and $(\hat{\beta},\hat{\phi})$ a gauge homotopy from β to γ we set

$$(\hat{\beta},\hat{\phi})\Box_\rho(\hat{\alpha},\hat{\varphi}) := (\hat{\beta}\Box_\rho\hat{\alpha}, \hat{\phi}\Box_\rho\hat{\varphi})$$

where

$$(\hat{\beta}\Box_\rho\hat{\alpha})(t) \;:=\; \begin{cases}\alpha_{2\rho(t)} & 0\le t\le 1/3\\ \alpha_1 & 1/3\le t\le 2/3\\ \beta_{(2\rho(t)-1)} & 2/3\le t\le 1\end{cases} \quad \text{and}$$

$$(\hat{\phi}\Box_\rho\hat{\varphi})(t) \;:=\; \begin{cases}\varphi_{2\rho(t)} & 0\le t\le 1/3\\ \varphi_1 & 1/3\le t\le 2/3\\ \phi_{(2\rho(t)-1)}\circ\phi_1 & 2/3\le t\le 1\end{cases} \quad \text{respectively.}$$

Observe that $(\hat{\beta},\hat{\phi})\Box_\rho(\hat{\alpha},\hat{\varphi})$ is a gauge homotopy from α to γ.

$\square$

DEFINITION 2.23. Let (E,Π) be a coisotropic vector bundle. The *moduli space of geometric Maurer–Cartan elements* $\mathcal{N}(E,\Pi)$ of (E,Π) is the set of equivalence classes of elements of $\mathcal{D}_{\mathrm{geo}}(E,\Pi)$ with respect to the equivalence relation $\sim_G$.

REMARK 2.24. The maps

$$L_{\mathrm{geo}}:\mathcal{D}_{\mathrm{geo}}(E,\Pi)\to\mathcal{C}(E,\Pi),\quad \underline{L}:\underline{\mathrm{Inn}}(BFV(E))\to\underline{\mathrm{Ham}}(E,\Pi)$$

introduced in Corollary 2.19 and Remark 2.13 yield a map $\hat{L}_{\mathrm{geo}}$ from the set of gauge homotopies of (E, Π) to the set of Hamiltonian homotopies of (E, Π) respectively:

$$\hat{L}_{\mathrm{geo}}(\hat{\beta}, \hat{\phi}) := (L_{\mathrm{geo}}(\hat{\beta}), \underline{L}(\hat{\phi})).$$

Observe that the vanishing locus of

$$\Omega^0 + \beta_t^0 = \Omega^0 + \phi_t \cdot \beta^0 = \phi_t(\Omega^0 + \beta^0)$$

is the image of the vanishing locus of $\Omega^0 + \beta^0$ under $\underline{L}(\phi_t)$, i.e.

$$S_{-L_{\mathrm{geo}}(\beta_t)} := \{(x, -L_{\mathrm{geo}}(\beta_t)(x)) : x \in S\} = \underline{L}(\phi_t)(S_{-L_{\mathrm{geo}}(\beta_0)})$$

and consequently $\hat{L}_{\mathrm{geo}}(\hat{\beta}, \hat{\phi})$ is in fact a Hamiltonian homotopy from $L_{\mathrm{geo}}(\beta_0)$ to $L_{\mathrm{geo}}(\beta_1)$.

Theorem 2.25. *Let (E, Π) be a coisotropic vector bundle. The map*

$$L_{\mathrm{nor}} : \mathcal{D}_{\mathrm{geo}}(E, \Pi) \to \mathcal{C}(E, \Pi)$$

from Corollary 2.19 induces a bijection

$$[L] : \mathcal{N}(E, \Pi) \xrightarrow{\cong} \mathcal{M}(E, \Pi).$$

Proof. Consider the surjective map

$$B : \mathcal{D}_{\mathrm{geo}}(E, \Pi) \xrightarrow{L_{\mathrm{geo}}} \mathcal{C}(E, \Pi) \to \mathcal{M}(E, \Pi).$$

We claim that two geometric Maurer–Cartan elements α and β of (E, Π) have the same image under B if and only if there is a gauge homotopy from α to β. Consequently B induces a bijection

$$[L] : \mathcal{N}(E, \Pi) \xrightarrow{\cong} \mathcal{M}(E, \Pi).$$

Given a gauge homotopy $(\hat{\alpha}, \hat{\phi})$ from α to β, $\hat{L}_{\mathrm{geo}}(\hat{\alpha}, \hat{\phi})$ is a gauge homotopy from $L_{\mathrm{geo}}(\alpha)$ to $L_{\mathrm{geo}}(\beta)$, hence the images of α and β under B coincide.

On the other hand assume that the class of $L_{\mathrm{geo}}(\alpha)$ equals to class of $L_{\mathrm{geo}}(\beta)$ in $\mathcal{M}(E, \Pi)$, i.e. there is a Hamiltonian homotopy $(-\hat{\mu}, \hat{\varphi})$ from $L_{\mathrm{geo}}(\alpha)$ to $L_{\mathrm{geo}}(\beta)$. We have to prove that this lifts to a gauge homotopy from α to β. Consider the smooth one-parameter family of coisotropic sections

$$-\hat{\mu} \in \Gamma(\wedge(\mathcal{E}_{[0,1]} \oplus \mathcal{E}^*_{[0,1]})).$$

In the proof of Theorem 1.13 we proved that for every $t \in [0, 1]$ there is a normalized Maurer–Cartan element ξ_t of (E, Π) such that $L_{\mathrm{nor}}(\xi_t) = -\mu_t$. Now we need the adapt the construction in the proof of Theorem 1.13 such that $(\xi_t)_{t\in[0,1]}$ is a smooth one-parameter family of sections of $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$. Recall Remark 1.14: for fixed $t \in [0, 1]$ one obtains a differential $\delta[\mu_t] := [\Omega^0 + p^*(\mu_t), \cdot]_G$ and the homotopy h constructed in the proof of Theorem 2.13 in Chapter 4 yields a homotopy $h[\mu_t]$ for $[\Omega^0 + p^*(\mu_t), \cdot]_G$ such that

$$[h[\mu_t], \delta[\mu_t]] = h[\mu_t] \circ \delta[\mu_t] + \delta[\mu_t] \circ h[\mu_t] = \mathrm{id} - (\wedge i^*_{\mu_t}) \circ (\wedge p^*_{\mu_t})$$

holds where $\wedge i^*_{\mu_t} : BFV(E) \to \Gamma(\wedge E)$ is given by projecting to the ghost-momentum degree 0 component and evaluation at $S_{-\mu_t}$ and $\wedge p^*_{\mu_t} : \Gamma(\wedge E) \to BFV(E)$ is given by extension of sections of $\wedge E$ to sections of $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$ that are constant along the fibres of $E \to S \cong S_{-\mu_t}$. These data can be assembled into

$$\begin{array}{lll} \text{a differential} & \hat{\delta} : \underline{BFV}(E) \to \underline{BFV}(E)[1], & (\gamma_t)_{t\in[0,1]} \mapsto (\delta[\mu_t](\gamma_t))_{t\in[0,1]}, \\ \text{a homotopy} & \hat{h} : \underline{BFV}(E) \to \underline{BFV}(E)[-1], & (\gamma_t)_{t\in[0,1]} \mapsto (h[\mu_t](\gamma_t))_{t\in[0,1]}, \\ \text{an inclusion} & \hat{p} : \Gamma(\wedge E_{[0,1]}) \to \underline{BFV}(E), & (\gamma_t)_{t\in[0,1]} \mapsto (\wedge p^*_{\mu_t}(\gamma_t))_{t\in[0,1]} \quad \text{and} \\ \text{a projection} & \hat{i} : \underline{BFV}(E) \to \Gamma(\wedge E_{[0,1]}), & (\gamma_t)_{t\in[0m1]} \mapsto (\wedge i^*_{\mu_t}(\gamma_t))_{t\in[0,1]} \end{array}$$

such that

$$[\hat{h}, \hat{\delta}] = \hat{h} \circ \hat{\delta} + \hat{\delta} \circ \hat{h} = \mathrm{id} - \hat{i} \circ \hat{p}$$

holds. In particular a cocycle $(\gamma_t)_{t\in[0,1]}$ of $(\underline{BFV}(E), \hat{\delta})$ is a coboundary if it is concentrated in positive ghost-momentum degrees or if the all restrictions $\gamma_t|_{S_{-\mu_t}}$ vanish. Following Remark 1.14 and the proof of Theorem 2.13 in Chapter 4 this allows us to construct a smooth one-parameter family $\hat{\xi}$ of normalized Maurer–Cartan element of (E, Π) such that the component of ξ_t in $\Gamma(\mathcal{E})$ is equal to $p^*(\mu_t)$.

Denote the lift $\underline{R}(\hat{\varphi})$ of the smooth one-parameter family of Hamiltonian diffeomorphisms $\hat{\varphi}$ by $\hat{\phi}$ and set

$$\hat{\chi} := \hat{\phi}^{-1} \cdot \hat{\xi},$$

i.e. $\chi_t := \phi_t^{-1} \cdot \xi_t$. By construction $\hat{\chi}$ is a smooth one-parameter family of Maurer–Cartan elements whose component $\hat{\chi}^0$ in $\Gamma(\mathcal{E}_{[0,1]})$ vanishes at $S_{-\mu_0}$. However it might not be a smooth one-parameter family of *geometric* Maurer–Cartan elements.

Using the automorphism of the bigraded algebra $\hat{\phi}_{\mu_0} : BFV(E) \to BFV(E)$ constructed in the proof of Theorem 1.13 we obtain a smooth one-parameter family of Maurer–Cartan elements $\hat{\delta} := \hat{\phi}_{\mu_0}(\Omega + \hat{\chi})$ – with respect to a BFV-bracket $[\cdot,\cdot]'_{BFV}$ for another Poisson bivector field Π_{μ_0} on E – whose component $\hat{\delta}^0$ in $\Gamma(\mathcal{E})$ vanishes on S. Furthermore

$$\delta^0_0 = \hat{\phi}_{\mu_0}(\Omega^0 + p^*(\mu)) = \Omega^0$$

and the graph of δ^0_t intersects the zero section of $\mathcal{E} \to E$ transversally for all $t \in [0,1]$ since $(\Omega^0 + \xi^0_t)$ and hence $(\Omega^0 + \chi^0_t)$ do so. Consequently we can apply Proposition 9 in Chapter 6, i.e. there is a smooth one-parameter family $\hat{A}$ of sections of $GL_+(\mathcal{E})$ such that

$$A_t(\Omega^0) = \delta^0_t$$

holds for arbitrary $t \in [0,1]$. By Lemma 2.12 the smooth one-parameter family

$$\left(a_s := -\frac{d}{dt}|_{t=s}(A_t) \circ A_s^{-1}\right)_{t\in[0,1]} \in \Gamma(\mathrm{End}(\mathcal{E})_{[0,1]}) \cong \Gamma(\mathcal{E}_{[0,1]} \otimes \mathcal{E}^*_{[0,1]})$$

integrates to a smooth one-parameter family $\hat{\Lambda}$ of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]'_{BFV})$. Hence

$$\psi_t := \hat{\phi}_{\mu_0}^{-1} \circ \Lambda_t \circ \hat{\phi}_{\mu_0}$$

is a smooth one-parameter family of inner automorphisms of (E, Π) satisfying

$$\begin{aligned}
\psi_t \cdot \chi_0 + BFV_{\geq 1}(E) &= \psi_t(\Omega + \chi_0) - \Omega + BFV_{\geq 1}(E) \\
&= (\hat{\phi}_{\mu_0}^{-1} \circ \Lambda_t \circ \hat{\phi}_{\mu_0})(\Omega + \chi_0) - \Omega^0 + BFV_{\geq 1}(E) \\
&= (\hat{\phi}_{\mu_0}^{-1} \circ \Lambda_t)(\delta_0^0) - \Omega^0 + BFV_{\geq 1}(E) \\
&= \hat{\phi}_{\mu_0}^{-1}(\delta_t^0) - \Omega^0 + BFV_{\geq 1}(E) \\
&= \Omega^0 + \chi_t^0 - \Omega^0 + BFV_{\geq 1}(E) \\
&= \chi_t^0 + BFV_{\geq 1}(E),
\end{aligned}$$

and $\hat{\psi} \cdot \chi_0$ is a smooth one-parameter family of Maurer–Cartan elements of (E, Π).

We calculate

$$\begin{aligned}
(\phi_t \circ \psi_t) \cdot \chi_0 + BFV_{\geq 1}(E) &= (\phi_t \circ \psi_t) \cdot \chi_0^0 + BFV_{\geq 1}(E) \\
&= \phi_t \cdot \chi_t^0 + BFV_{\geq 1}(E) \\
&= \xi_t^0 + BFV_{\geq 1}(E).
\end{aligned}$$

Because $\hat{\xi}$ is a smooth one-parameter family of normalized and hence *geometric* Maurer–Cartan elements of (E, Π), so is $(\hat{\phi} \circ \hat{\psi}) \cdot \chi_0$ and consequently

$$((\hat{\phi} \circ \hat{\psi}) \cdot \chi_0, \hat{\phi} \circ \hat{\psi})$$

is a gauge homotopy from χ_0 to $(\phi_1 \circ \psi_1) \cdot \chi_0$, i.e.

$$\chi_0 \sim_G (\phi_1 \circ \psi_1) \cdot \chi_0.$$

By construction

$$L_{\text{geo}}(\chi_0) = -\mu_0 = L_{\text{geo}}(\alpha) \quad \text{and} \quad L_{\text{geo}}((\varphi_1 \circ \psi_1) \cdot \chi_0) = -\mu_1 = L_{\text{geo}}(\beta)$$

hold. Since α, β, χ_0 and $(\phi_1 \circ \psi_1) \cdot \chi_0$ are geometric Maurer–Cartan elements we can apply Corollary 2.19 and find $\hat{\zeta}$ and $\hat{\eta} \in \underline{\text{Inn}}_{\geq 1}(BFV(E))$ satisfying

$$\zeta_1 \cdot \alpha = \chi_0 \quad \text{and} \quad \eta_1 \cdot ((\phi_1 \circ \psi_1) \cdot \chi_0) = \beta \quad \text{respectively.}$$

In particular $(\hat{\zeta} \cdot \chi_0, \hat{\zeta})$ is a gauge homotopy from α to χ_0 and $(\hat{\eta} \cdot ((\phi_1 \circ \psi_1) \cdot \chi_0), \hat{\eta})$ is a gauge homotopy from $(\phi_1 \circ \psi_1) \cdot \chi_0$ to β. Finally

$$\alpha \sim_G \chi_0 \sim_G (\phi_1 \circ \psi_1) \cdot \chi_0 \sim_G \beta$$

yields $\alpha \sim_G \beta$. $\square$

3. Deformation Groupoids

DEFINITION 3.1. A *gluing function* is a smooth function $\rho : [0,1] \to [0,1]$ satisfying

(i) $\rho(0) = 0$ and $\rho(1) = 1$,
(ii) ρ is equal to $1/2$ on $[1/3, 2/3]$ and
(iii) the restriction of ρ to $]0, 1/3[$ and $]2/3, 1[$ are diffeomorphisms to $\rho(]0, 1/3[)$ and $\rho(]1/3, 1[)$ respectively.

REMARK 3.2. The existence of a gluing function is proved in Lemma 12 in Chapter 6. In the proofs of Lemma 2.5 and Lemma 2.22 an operation $\square_\rho$ on composable pairs of Hamiltonian and gauge homotopies respectively was constructed. We refer to this operation as the *composition* (of Hamiltonian homotopies and gauge homotopies respectively) *with respect to* $\square_\rho$. The existence of $\square_\rho$ implies transitivity of the relations $\sim_H$ and $\sim_G$.

Given a coisotropic vector bundle (E, Π), we want to equip the set of coisotropic sections $\mathcal{C}(E, \Pi)$ and the set of geometric Maurer–Cartan elements $\mathcal{D}_{\mathrm{geo}}(E, \Pi)$ with the structure of groupoids such that the sets of isomorphism classes of objects are isomorphic to the moduli space of coisotropic sections $\mathcal{M}(E, \Pi)$ and the moduli space of geometric Maurer–Cartan elements $\mathcal{N}(E, \Pi)$ respectively. The isomorphism between $\mathcal{M}(E, \Pi)$ and $\mathcal{N}(E, \Pi)$ established Theorem 2.25 generalizes to a surjective morphism between these groupoids and we will give an explicit description of its kernel.

DEFINITION 3.3. Given a coisotropic vector bundle (E, Π) a *smooth two-parameter family of Hamiltonian diffeomorphisms* $\hat{\Phi}$ is a smooth map

$$\hat{\Phi} : E \times [0,1]^2 \to E, \quad (e, t, s) \mapsto \Phi(t, s)(e)$$

satisfying the following conditions:

(i) $\Phi(0, s)$ is the identity for all $s \in [0, 1]$ and
(ii) there is a smooth function

$$F : E \times [0,1]^2 \to \mathbb{R}, \quad (e, t, s) \mapsto F(t, s)(e)$$

such that $\Phi(\cdot, s)$ is the smooth one-parameter family of Hamiltonian diffeomorphisms generated by $F(\cdot, s)$, i.e.

$$\frac{d}{d\tau}|_{\tau=t}\Phi(\tau, s) = X_{F(t,s)(\cdot)}|_{\Phi(t,s)(x)}$$

holds for all $(t, s) \in [0, 1]^2$.

An *isotopy of Hamiltonian homotopies* of (E, Π) is a pair $(\hat{\mu}, \hat{\Phi})$ where

(a) $\hat{\mu}$ is a smooth $[0,1]^2$-family of sections of E whose restriction $\mu_{t,s}$ is a coisotropic section of (E, Π) for all $(t, s) \in [0, 1]^2$,
(b) $\hat{\Phi}$ is a smooth two-parameter family of Hamiltonian diffeomorphisms

such that

(a') both restrictions of $\hat{\mu}$ to $E \times \{0\} \times \{s\}$ and $E \times \{1\} \times \{s\}$ respectively are constant in s and

(b') the graph $S_{\mu_{(t,s)}}$ of $\mu_{(t,s)}$ is equal to the image of $S_{\mu_{(0,s)}}$ under $\Phi(t,s)$ for arbitrary $(t,s) \in [0,1]^2$.

Given an isotopy of Hamiltonian homotopies $(\hat{\mu}, \hat{\Phi})$ we say that it is an isotopy of Hamiltonian homotopies *from* the Hamiltonian homotopy $(\hat{\mu}_{(t,0)}, \Phi_{(t,0)})$ *to* the Hamiltonian homotopy $(\hat{\mu}_{(t,1)}, \Phi_{(t,1)})$. We also refer to the pair consisting of these Hamiltonian homotopies as the *vertical boundary of* $(\hat{\mu}, \hat{\Phi})$. The *horizontal boundary* is the pair consisting of the Hamiltonian homotopies $(\mu_{(0,s)}, (\mathrm{id})_{s\in[0,1]})$ and $(\mu_{(1,s)}, (\mathrm{id})_{s\in[0,1]})$.

Definition 3.4. Let (E, Π) be a coisotropic vector bundle. We define a relation $\simeq_H$ on the set of Hamiltonian homotopies of (E, Π) by

$(\hat{\mu}, \hat{\varphi}) \simeq_H (\hat{\nu}, \hat{\phi}) :\Leftrightarrow$

there is an isotopy of Hamiltonian homotopies from $(\hat{\mu}, \hat{\varphi})$ to $(\hat{\nu}, \hat{\phi})$.

Lemma 3.5. (a) *The relation $\simeq_H$ on the set of Hamiltonian homotopies of (E, Π) is an equivalence relation.*

(b) *Let ρ and ρ' be two gluing functions. Then the compositions of Hamiltonian homotopies with respect to ρ and ρ' coincide up to $\simeq_H$.*

(c) *The Hamiltonian homotopies*

$$\mathrm{id}_{\mu_0} \Box_\rho (\hat{\mu}, \hat{\varphi}) \quad \textit{and} \quad (\hat{\mu}, \hat{\varphi}) \Box_\rho \mathrm{id}_{\mu_1}$$

are equivalent to $(\hat{\mu}, \hat{\varphi})$ under $\simeq_H$.

(d) *The Hamiltonian homotopies*

$$(\hat{\mu}, \hat{\varphi})^{-1} \Box_\rho (\hat{\mu}, \hat{\varphi}) \quad \textit{and} \quad (\hat{\mu}, \hat{\varphi}) \Box_\rho (\hat{\mu}, \hat{\varphi})^{-1}$$

are equivalent to id_{μ_0} under $\simeq_H$.

(e) *Composition with respect to ρ descends to the set of equivalence classes of Hamiltonian homotopies modulo $\simeq_H$ and is associative there.*

Proof. (a) Any Hamiltonian homotopy $(\hat{\mu}, \hat{\varphi})$ comes along with an isotopy of Hamiltonian homotopies

$$\mathrm{id}_{(\hat{\mu}, \hat{\varphi})} := ((\hat{\mu})_{s\in[0,1]}, (\hat{\varphi})_{s\in[0,1]})$$

from $(\hat{\mu}, \hat{\varphi})$ to $(\hat{\mu}, \hat{\varphi})$.

Given an isotopy of Hamiltonian homotopies $(\hat{\mu}, \hat{\Phi})$ from $(\mu_{(t,0)}, \Phi_{(t,0)})$ to $(\mu_{(t,1)}, \Phi_{(t,0)})$, the pair

$$(\hat{\mu}, \hat{\Phi})^{-1} := (\mu_{(t,1-s)}, \Phi_{(t,1-s)})$$

is an isotopy of Hamiltonian homotopies from $(\mu_{(t,1)}, \Phi_{(t,0)})$ to $(\mu_{(t,0)}, \Phi_{(t,0)})$.

Let $(\hat{\mu}, \hat{\Phi})$ be an isotopy of Hamiltonian homotopies from $(\mu_{(t,0)}, \Phi_{(t,0)})$ to $(\mu_{(t,1)}, \Phi_{(t,1)})$ and $(\hat{\nu}, \hat{\Psi})$ an isotopy of Hamiltonian homotopies from

$(\nu_{(t,0)}, \Psi_{(t,0)})$ to $(\nu_{(t,1)}, \Psi_{(t,1)})$ and suppose that $(\mu_{(t,1)}, \Phi_{(t,1)}) = (\nu_{(t,0)}, \Psi_{(t,0)})$ holds. Choose a gluing function ρ and define the *vertical composition*

$$(\hat{\nu}, \hat{\Psi})\Box^V_\rho(\hat{\mu}, \hat{\Phi}) := (\hat{\nu}\Box^V_\rho\hat{\mu}, \hat{\Psi}\Box^V\hat{\Phi})$$

of $(\hat{\mu}, \hat{\Phi})$ and $(\hat{\nu}, \hat{\Psi})$ *with respect to* ρ by

$$(\hat{\nu}\Box^V_\rho\hat{\mu})(t,s) \quad := \quad \begin{cases} \mu_{(t,2\rho(s))} & 0 \le s \le 1/3 \\ \mu_{(t,1)} & 1/3 \le s \le 2/3 \\ \nu_{(t,2\rho(s)-1)} & 2/3 \le s \le 1 \end{cases} \quad \text{and}$$

$$(\hat{\Psi}\Box^V_\rho\hat{\Phi})(t,s) \quad := \quad \begin{cases} \Phi_{(t,2\rho(s))} & 0 \le s \le 1/3 \\ \Phi_{(t,1)} & 1/3 \le s \le 2/3 \\ \Psi_{(t,2\rho(s)-1)} & 2/3 \le s \le 1 \end{cases} \quad \text{respectively.}$$

Observe that $(\hat{\nu}, \hat{\Psi})\Box^V_\rho(\hat{\mu}, \hat{\Phi})$ is an isotopy of Hamiltonian homotopies from $(\mu_{(t,0)}, \Phi_{(t,0)})$ to $(\nu_{(t,1)}, \Psi_{(t,1)})$.

(b) Let $(\hat{\mu}, \hat{\phi})$ be a Hamiltonian homotopy from μ to ν and suppose ρ and ρ' are two gluing functions. Choose a smooth function $\sigma : [0,1] \to [0,1]$ that is zero on $[0,1/3]$, equal to 1 on $[2/3,1]$ and the restriction to $]1/3,2/3[$ is a diffeomorphism to $\sigma(]1/3,2/3[)$. Existence of such a function is proved similar to the existence of a gluing function. The smooth one-parameter family of gluing functions

$$\hat{\rho}(s) := (1-\sigma(s))\rho + \sigma(s)\rho'$$

defines an isotopy of Hamiltonian homotopies

$$(\hat{\mu}, \hat{\phi})\Box_{\hat{\rho}(s)}(\hat{\nu}, \hat{\psi})$$

from $(\hat{\mu}, \hat{\phi})$ to $(\hat{\nu}, \hat{\psi})$.

(c) The Hamiltonian homotopy $(\hat{\mu}, \hat{\varphi})\Box_\rho \operatorname{id}_{\mu_1}$ is given by

$$\begin{cases} \mu_{2\rho(t)} & 0 \le t \le 1/3 \\ \mu_1 & 1/3 \le t \le 1 \end{cases} \quad \text{and} \quad \begin{cases} \varphi_{2\rho(t)} & 0 \le t \le 1/3 \\ \varphi_1 & 1/3 \le t \le 1 \end{cases}.$$

Choose a smooth function $\sigma : [0,1] \to [0,1]$ that is zero on $[0,1/3]$, equal to 1 on $[2/3,1]$ and the restriction to $]1/3,2/3[$ is a diffeomorphism to $\sigma(]1/3,2/3[)$. Setting $((\hat{\mu}, \hat{\varphi})\Box_\rho \operatorname{id}_{\mu_1}) \circ g_s(t)$ with

$$g_s(t) := (1 - \frac{2}{3}(1-\sigma(s)))t$$

yields an isotopy of Hamiltonian homotopies from $(\hat{\mu}, \hat{\varphi})\Box_\rho \operatorname{id}_{\mu_1}$ to $(\mu_{\alpha(t)}, \varphi_{\alpha(t)})$ where α is a diffeomorphism of $[0,1]$ relative to the boundary. Now

$$(\mu_{((1-\sigma(s))\alpha(t)+\sigma(s)t)}, \varphi_{((1-\sigma(s))\alpha(t)+\sigma(s)t)})$$

is an isotopoy of Hamiltonian homotopies from $(\mu_{\alpha(t)}, \varphi_{\alpha(t)})$ to $(\hat{\mu}, \hat{\varphi})$. Since $\simeq_H$ is an equivalence relation we obtain $(\hat{\mu}, \hat{\varphi})\Box_\rho \operatorname{id}_{\mu_1} \simeq_H (\hat{\mu}, \hat{\varphi})$.

Similiarly one finds an isotopy of Hamiltonian homotopies from $id_{\mu_0}\Box_\rho(\hat{\mu},\hat{\varphi})$ to $(\hat{\mu},\hat{\varphi})$.

(d) The Hamiltonian homotopy $(\hat{\mu},\hat{\varphi})^{-1}\Box_\rho(\hat{\mu},\hat{\varphi})$ is given by

$$\begin{cases}\mu_{2\rho(t)} & 0\le t\le 1/3\\ \mu_1 & 1/3\le t\le 2/3\\ \mu_{2(1-\rho(t))} & 2/3\le t\le 1\end{cases}\quad\text{and}\quad\begin{cases}\varphi_{2\rho(t)} & 0\le t\le 1/3\\ \varphi_1 & 1/3\le t\le 2/3\\ \varphi_{2(1-\rho(t))} & 2/3\le t\le 1\end{cases}.$$

Choose a smooth function $\sigma:[0,1]\to[0,1]$ that is zero on $[0,1/3]$, equal to 1 on $[2/3,1]$ and the restriction to $]1/3,2/3[$ is a diffeomorphism to $\sigma(]1/3,2/3[)$. Then

$$\begin{cases}\mu_{2\rho(t)(1-\sigma(s))} & 0\le t\le 1/3\\ \mu_{(1-\sigma(s))} & 1/3\le t\le 2/3\\ \mu_{2(1-\rho(t))(1-\sigma(s))} & 2/3\le t\le 1\end{cases},\begin{cases}\varphi_{2\rho(t)(1-\sigma(s))} & 0\le t\le 1/3\\ \varphi_{(1-\sigma(s))} & 1/3\le t\le 2/3\\ \varphi_{2(1-\rho(t))(1-\sigma(s))} & 2/3\le t\le 1\end{cases}$$

defines an isotopy of Hamiltonian homotopies from $(\hat{\mu},\hat{\varphi})^{-1}\Box_\rho(\hat{\mu},\hat{\varphi})$ to id_{μ_0}. Similarly one finds an isotopy of Hamiltonian homotopies from $(\hat{\mu},\hat{\varphi})\Box_\rho(\hat{\mu},\hat{\varphi})^{-1}$ to id_{μ_0}.

(e) Let $(\hat{\mu},\hat{\Phi})$ and $(\hat{\nu},\hat{\Psi})$ be isotopies of Hamiltonian homotopies such that $\Phi_{(1,s)}=\Psi_{(0,s)}$. Choose a gluing function ρ and define *horizontal composition*

$$(\hat{\nu},\hat{\Psi})\Box^H_\rho(\hat{\mu},\hat{\Phi}):=(\hat{\nu}\Box^H_\rho\hat{\mu},\hat{\Psi}\Box^H_\rho\hat{\Phi})$$

of $(\hat{\mu},\hat{\Phi})$ and $(\hat{\nu},\hat{\Psi})$ *with respect to* ρ by

$$(\hat{\nu}\Box^H_\rho\hat{\mu})(t,s)\;:=\;\begin{cases}\mu_{(2\rho(t),s)} & 0\le t\le 1/3\\ \mu_{(1,s)} & 1/3\le t\le 2/3\\ \nu_{(2\rho(t)-1,s)} & 2/3\le s\le 1\end{cases}\quad\text{and}$$

$$(\hat{\Psi}\Box^H_\rho\hat{\Phi})(t,s)\;:=\;\begin{cases}\Phi_{(2\rho(t),s)} & 0\le s\le 1/3\\ \Phi_{(1,s)} & 1/3\le s\le 2/3\\ \Psi_{(2\rho(t)-1,s)}\circ\Phi_1 & 2/3\le s\le 1\end{cases}.$$

Observe that $(\hat{\nu},\hat{\Psi})\Box^H_\rho(\hat{\mu},\hat{\Phi})$ is an isotopy of Hamiltonian homotopies from $(\nu_{(t,0)},\Psi_{(t,0)})\Box_\rho(\mu_{(t,0)},\Phi_{(t,0)})$ to $(\nu_{(t,1)},\Psi_{(t,1)})\Box_\rho(\mu_{(t,1)},\Phi_{(t,1)})$. This implies that $\Box_\rho$ descends the set of equivalence classes with respect to $\simeq_H$.

Suppose $(\hat{\mu},\hat{\phi})$, $(\hat{\nu},\hat{\psi})$ and $(\hat{\tau},\hat{\varphi})$ are Hamiltonian homotopies such that

$$\phi_1=\psi_0\quad\text{and}\quad\psi_1=\varphi_0$$

hold. We want to prove

$$A:=(\hat{\tau},\hat{\varphi})\Box_\rho\left((\hat{\nu},\hat{\psi})\Box_\rho(\hat{\mu},\hat{\phi})\right)\simeq_H\left((\hat{\tau},\hat{\varphi})\Box_\rho(\hat{\nu},\hat{\psi})\right)\Box_\rho(\hat{\mu},\hat{\phi})=:B$$

where ρ is an arbitrary gluing function. Let θ_s, ϑ_s be two smooth one-parameter families of diffeomorphisms of $[0,1]$ relative to the boundary that start at the identity such that

(i) the diffeomorphism θ_1 maps

$$\begin{array}{llllll} [0,1/5], & [1/5,2/5], & [2/5,3/5], & [3/5,4/5], & [4/5,1] \text{ to} \\ [0,1/9], & [1/9,2/9], & [2/9,1/3], & [1/3,2/3], & [2/3,1]., \end{array}$$

(ii) the diffeomorphism ϑ_1 maps

$$\begin{array}{llllll} [0,1/5], & [1/5,2/5], & [2/5,3/5], & [3/5,4/5], & [4/5,1] \text{ to} \\ [0,1/3], & [1/3,2/3], & [2/3,7/9], & [7/9,8/9], & [8/9,1]. \end{array}$$

The existence of such smooth one-parameter families of diffeomorphisms of $[0,1]$ is proved in Lemma 13 in Chapter 6. We consider

$$A(t,s) := \Big(\big((\hat{\tau} \Box_\rho (\hat{\nu} \Box_\rho \hat{\mu}))(\theta_s(t)) \big), \big((\hat{\varphi} \Box_\rho (\hat{\psi} \Box_\rho \hat{\phi}))(\theta_s(t)) \big) \Big)$$

and

$$B(t,s) := \Big(\big(((\hat{\tau} \Box_\rho \hat{\nu}) \Box_\rho \hat{\mu})(\vartheta_s(t)) \big), \big(((\hat{\varphi} \Box_\rho \hat{\psi}) \Box_\rho \hat{\phi})(\vartheta_s(t)) \big) \Big).$$

These are two isotopies of Hamiltonian homotopies from A to $A(t,1)$ and from B to $B(t,1)$ respectively. Observe that $A(t,1)$ is given by

$$\begin{cases} \mu_{\alpha_1(5t)} & 0 \le t \le 1/5 \\ \mu_1 & 1/5 \le t \le 2/5 \\ \nu_{\alpha_2(5t-2)} & 2/5 \le t \le 3/5 \\ \nu_1 & 3/4 \le t \le 4/5 \\ \tau_{\alpha_3(5t-4)} & 4/5 \le t \le 1 \end{cases}, \begin{cases} \phi_{\alpha_1(5t)} & 0 \le t \le 1/5 \\ \phi_1 & 1/5 \le t \le 2/5 \\ \psi_{\alpha_2(5t-2)} \circ \phi_1 & 2/5 \le t \le 3/5 \\ \psi_1 \circ \phi_1 & 3/4 \le t \le 4/5 \\ \varphi_{\alpha_3(5t-4)} \circ \psi_1 \circ \phi_1 & 4/5 \le t \le 1 \end{cases}$$

where α_1, α_2 and α_3 are diffeomorphisms of $[0,1]$ relative to the boundary. Similarly $B(t,1)$ is given by

$$\begin{cases} \mu_{\beta_1(5t)} & 0 \le t \le 1/5 \\ \mu_1 & 1/5 \le t \le 2/5 \\ \nu_{\beta_2(5t-2)} & 2/5 \le t \le 3/5 \\ \nu_1 & 3/4 \le t \le 4/5 \\ \tau_{\beta_3(5t-4)} & 4/5 \le t \le 1 \end{cases}, \begin{cases} \phi_{\beta_1(5t)} & 0 \le t \le 1/5 \\ \phi_1 & 1/5 \le t \le 2/5 \\ \psi_{\beta_2(5t-2)} \circ \phi_1 & 2/5 \le t \le 3/5 \\ \psi_1 \circ \phi_1 & 3/4 \le t \le 4/5 \\ \varphi_{\beta_3(5t-4)} \circ \psi_1 \circ \phi_1 & 4/5 \le t \le 1 \end{cases}.$$

Now we choose a smooth function $\sigma : [0,1] \to [0,1]$ that is zero on $[0,1/3]$, equal to 1 on $[2/3,1]$ and the restriction to $]1/3,2/3[$ is a diffeomorphism to $\sigma(]1/3, 2/3[)$ and define $C(t,s)$ to be

$$\begin{cases} \mu_{(1-\sigma(s))\alpha_1(5t)+\sigma(s)\beta_1(5t)} & 0 \le t \le 1/5 \\ \mu_1 & 1/5 \le t \le 2/5 \\ \nu_{(1-\sigma(s))\alpha_2(5t-2)+\sigma(s)\beta_2(5t-2)} & 2/5 \le t \le 3/5 \\ \nu_1 & 3/4 \le t \le 4/5 \\ \tau_{(-1\sigma(s))\alpha_3(5t-4)+\sigma(s)\beta_3(5t-4)} & 4/5 \le t \le 1 \end{cases},$$

$$\begin{cases} \phi_{(1-\sigma(s))\alpha_1(5t)+\sigma(s)\beta_1(5t)} & 0 \le t \le 1/5 \\ \phi_1 & 1/5 \le t \le 2/5 \\ \psi_{(1-\sigma(s))\alpha_2(5t-2)+\sigma(s)\beta_2(5t-2)} \circ \phi_1 & 2/5 \le t \le 3/5 \\ \psi_1 \circ \phi_1 & 3/4 \le t \le 4/5 \\ \varphi_{(-1\sigma(s))\alpha_3(5t-4)+\sigma(s)\beta_3(5t-4)} \circ \psi_1 \circ \phi_1 & 4/5 \le t \le 1 \end{cases}.$$

Observe that $C(t,s)$ is an isotopy of Hamiltonian homotopies from $A(t,1)$ to $B(t,1)$. This implies

$$A \simeq_H A(t,1) \simeq_H B(t,1) \simeq_H B$$

and hence $A \simeq_H B$, i.e. $\Box_\rho$ is associative up to $\simeq_H$.

$\Box$

Definition 3.6. A *(small) groupoid* $\mathcal{G}$ is a (small) category such that every morphism is invertible.

Definition 3.7. Given a coisotropic vector bundle (E,Π), the *groupoid of coisotropic sections* $\hat{\mathcal{C}}(E,\Pi)$ of (E,Π) is the small groupoid with

(a) the set of objects is the set of coisotropic sections $\mathcal{C}(E,\Pi)$ of (E,Π),
(b) the set of morphisms $\mathrm{Hom}(\mu,\nu)$ between two coisotropic sections μ and ν is the set of all Hamiltonian homotopies from μ to ν modulo $\simeq_H$ and
(c) the composition is induced from composition of Hamiltonian homotopies with respect to some gluing function.

Lemma 3.8. *Let (E,Π) be a coisotropic vector bundle. The set of isomorphism classes of objects of $\hat{\mathcal{C}}(E,\Pi)$ is the moduli space of coisotropic sections $\mathcal{M}(E,\Pi)$.*

Proof. This follows immediately from the definition of $\hat{\mathcal{C}}(E,\Pi)$, $\sim_H$ and $\mathcal{M}(E,\Pi)$. $\Box$

Remark 3.9. The groupoid $\hat{\mathcal{C}}(E,\Pi)$ is a "categorification" of $\mathcal{M}(E,\Pi)$. It seems very likely that it is the first level of a tower of such categorifications of $\mathcal{M}(E,\Pi)$. In fact it should be possible to understand $\hat{\mathcal{C}}(E,\Pi)$ as a truncation of a weak ∞-groupoid $\hat{\mathcal{C}}^\infty(E,\Pi)$ at its two-morphisms which are presumably given by isotopies of Hamiltonian homotopies.

DEFINITION 3.10. Let (E,Π) be a coisotropic vector bundle. Equip $\Gamma(\wedge(\mathcal{E}_{[0,1]^2}\oplus\mathcal{E}^*_{[0,1]^2})$ with the structure of a graded Poisson algebra given by the trivial extension the graded Poisson bracket $[\cdot,\cdot]_{BFV}$ on $BFV(E)$. We denote the restriction map

$$\Gamma(\wedge(\mathcal{E}_{[0,1]^2}\oplus\mathcal{E}^*_{[0,1]^2}))\to BFV(E),\quad \hat{\beta}\mapsto\beta|_{E\times\{t\}\times\{s\}}$$

by $\mathrm{ev}_{(t,s)}$.

A *smooth two-parameter family of inner automorphisms* $\hat{\Phi}$ is a morphism of graded Poisson algebras

$$\hat{\Phi}:(BFV(E),[\cdot,\cdot]_{BFV})\to(\Gamma(\wedge(\mathcal{E}_{[0,1]^2}\oplus\mathcal{E}^*_{[0,1]^2})),[\cdot,\cdot]_{BFV})$$

satisfying the following conditions:

(i) $\mathrm{ev}_{(0,s)}\circ\hat{\Phi}=(\mathrm{id})_{t\in[0,1]}$,
(ii) there is $\hat{\gamma}\in\Gamma(\wedge(\mathcal{E}_{[0,1]^2}\oplus\mathcal{E}^*_{[0,1]^2}))$ such that $\mathrm{ev}_{(\cdot,s)}\circ\hat{\Phi}$ is the smooth one-parameter family of inner automorphisms of $(BFV(E),[\cdot,\cdot]_{BFV})$ generated by $\mathrm{ev}_{(\cdot,s)}\circ\hat{\gamma}$, i.e.

$$\frac{d}{d\tau}|_{\tau=t}(\mathrm{ev}_{\tau,s}\circ\Phi)(\cdot)=-[\mathrm{ev}_{t,s}\circ\hat{\gamma},\mathrm{ev}_{t,s}\circ\Phi(\cdot)]_{BFV}$$

holds for all $(t,s)\in[0,1]^2$.

We abbreviate $\mathrm{ev}_{(t,s)}\circ\hat{\Phi}$ by $\Phi_{(t,s)}$.

An *isotopy of gauge homotopies* of (E,Π) is a pair $(\hat{\beta},\hat{\Phi})$ where

(a) $\hat{\beta}$ is a smooth $[0,1]^2$-family of sections of $\wedge(\mathcal{E}\otimes\mathcal{E}^*)$ whose restriction $\beta_{(t,s)}$ is a geometric Maurer–Cartan element for all $(t,s)\in[0,1]^2$,
(b) $\hat{\Phi}$ is a smooth two-parameter family of inner automorphisms of

$$(BFV(E),[\cdot,\cdot]_{BFV})$$

such that

(a') the restrictions $\beta_{(0,s)}$ and $\beta_{(1,s)}$ are constant in $s\in[0,1]$,
(b') $\beta_{(t,s)}=\Phi_{(t,s)}\cdot\beta_{(0,s)}$ holds for arbitrary $(t,s)\in[0,1]^2$.

Given an isotopy of gauge homotopies $(\hat{\beta},\hat{\Phi})$ we say that it is an isotopy of gauge homotopies *from* the gauge homotopy $(\beta_{(t,0)},\Phi_{(t,0)})$ *to* the gauge homotopy $(\beta(t,1),\Phi_{(t,1)})$. We also refer to the pair consisting of these gauge homotopies as the *vertical boundary* of $(\hat{\beta},\hat{\Phi})$. The *horizontal boundary* is the pair consisting of the gauge homotopies $(\beta_{(0,s)},(\mathrm{id})_{s\in[0,1]})$ and $(\beta_{(1,s)},(\mathrm{id})_{s\in[0,1]})$.

DEFINITION 3.11. Let (E,Π) be a coisotropic vector bundle. We define a relation $\simeq_G$ on the set of gauge homotopies by

$$(\hat{\alpha},\hat{\varphi})\simeq_G(\hat{\beta},\hat{\phi}):\Leftrightarrow$$
$$\text{there is an isotopy of gauge homotopies from }(\hat{\alpha},\hat{\varphi})\text{ to }(\hat{\beta},\hat{\phi}).$$

LEMMA 3.12. (a) *The relation $\simeq_G$ on the set of gauge homotopies of (E, Π) is an equivalence relation.*

(b) *Let ρ and ρ' be two gluing functions. Then the compositions of gauge homotopies with respect to ρ and ρ' coincide up to $\simeq_G$.*

(c) *The gauge homotopies*

$$\mathrm{id}_{\alpha_0} \Box_\rho (\hat{\alpha}, \hat{\varphi}) \quad \textit{and} \quad (\hat{\alpha}, \hat{\varphi}) \Box_\rho \mathrm{id}_{\alpha_1}$$

are equivalent to $(\hat{\alpha}, \hat{\varphi})$ under $\simeq_G$.

(d) *The gauge homotopies*

$$(\hat{\alpha}, \hat{\varphi})^{-1} \Box_\rho (\hat{\alpha}, \hat{\varphi}) \quad \textit{and} \quad (\hat{\alpha}, \hat{\varphi}) \Box_\rho (\hat{\alpha}, \hat{\varphi})^{-1}$$

are equivalent to id_{α_0} under $\simeq_G$.

(e) *Composition with respect to ρ descends to the set of equivalence classes of gauge homotopies modulo $\simeq_G$ and is associative there.*

PROOF. The proof can be copied mutatis mutandis from the proof of Lemma 3.5. In particular isotopies of gauge homotopies with matching boundary components can be composed *vertically* and *horizontally with respect to* a gluing function. $\Box$

DEFINITION 3.13. Given a coisotropic vector bundle (E, Π), the *groupoid of geometric Maurer–Cartan elements* $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ of (E, Π) is the small groupoid with

(a) the set of objects is the set of geometric Maurer–Cartan elements $\mathcal{D}_{\mathrm{geo}}(E, \Pi)$ of (E, Π),
(b) the set of morphisms $\mathrm{Hom}(\alpha, \beta)$ between two geometric Maurer–Cartan elements α and β is the set of all gauge homotopies form α to β modulo $\simeq_G$ and
(c) the composition is induced from the composition of gauge homotopies with respect to some gluing function.

LEMMA 3.14. *Let (E, Π) be a coisotropic vector bundle. The set of isomorphism classes of objects in $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ is the moduli space of geometric Maurer–Cartan elements $\mathcal{N}(E, \Pi)$.*

PROOF. This follows immediately from the definition of $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$, $\sim_G$ and $\mathcal{N}(E, \Pi)$. $\Box$

REMARK 3.15. The groupoid $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ is a "categorification" of $\mathcal{N}(E, \Pi)$. It seems very likely that it is the first level of a tower of such categorifications of $\mathcal{N}(E, \Pi)$. In fact it should be possible to understand $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ as a truncation of a weak ∞-groupoid $\hat{\mathcal{D}}^{\infty}_{\mathrm{geo}}(E, \Pi)$ at its two-morphisms which are presumably given by isotopies of gauge homotopies.

DEFINITION 3.16. A gauge homotopy $(\hat{\beta}, \hat{\phi})$ of a coisotropic vector bundle (E, Π) is *pure* if the smooth one-parameter family of inner automorphisms $\hat{\phi}$ lies in $\underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$, i.e. is generated by an element of $\underline{BFV}_{\geq 1}(E, \Pi)$.

A morphism of $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$ is *pure* if it can be represented by a pure gauge homotopy. We denote the set of pure morphisms between two geometric Maurer–Cartan elements α and β of (E,Π) by $\mathrm{Hom}_{\geq 1}(\alpha,\beta) \subset \mathrm{Hom}(\alpha,\beta)$.

Definition 3.17. Given a coisotropic vector bundle (E,Π), we define $\hat{\mathcal{D}}^{\geq 1}_{\mathrm{geo}}(E,\Pi)$ to be the full subgroupoid of $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$ whose set of objects is the set of geometric Maurer–Cartan elements $\mathcal{D}_{\mathrm{geo}}(E,\Pi)$ of (E,Π) and whose set of morphisms between two Maurer–Cartan elements α and β is $\mathrm{Hom}_{\geq 1}(\alpha,\beta)$.

Remark 3.18. It is easy to check that $\hat{\mathcal{D}}^{\geq 1}_{\mathrm{geo}}(E,\Pi)$ is a full subgroupoid:

(i) for every geometric Maurer–Cartan element α, the identity morphism id_α is pure,
(ii) the inverse of any pure morphism of $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$ is pure and
(iii) given two composable pure morphisms of $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$ their composition is pure again.

In Theorem 3.25 we will establish that $\hat{\mathcal{D}}^{\geq 1}_{\mathrm{geo}}(E,\Pi)$ is *normal*, i.e. given a morphism $f \in \mathrm{Hom}(\alpha,\beta)$ and a pure morphism $g \in \mathrm{Hom}^{\geq 1}(\beta,\beta)$, the morphism $f^{-1} \circ g \circ f$ is pure.

Lemma 3.19. *Let (E,Π) be a coisotropic vector bundle. The maps*

$$L_{\mathrm{geo}} \quad \textit{and} \quad \hat{L}_{\mathrm{geo}}$$

introduced in Corollary 2.19 and Remark 2.24 yield a morphism of groupoids

$$\mathcal{L}_{\mathrm{geo}} : \hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi) \to \hat{\mathcal{C}}(E,\Pi)$$

that is surjective on the sets of objects and on all morphism sets.

Proof. That $\mathcal{L}_{\mathrm{geo}}$ is a well-defined map from the set of morphisms from α to β in $\hat{\mathcal{D}}_{\mathrm{geo}}(E,\Pi)$ to the set of morphisms from $L_{\mathrm{geo}}(\alpha)$ to $L_{\mathrm{geo}}(\beta)$ in $\hat{\mathcal{C}}(E,\Pi)$ follows from the fact that the map $\hat{L}_{\mathrm{geo}}$ from the set of gauge homotopies to the set of Hamiltonian homotopies introduced in Remark 2.24 extends to a map from the set of isotopies of gauge homotopies to the set of isotopies of Hamiltonian homotopies.

Furthermore $\mathcal{L}_{\mathrm{geo}}$ is a morphism of groupoids because $\underline{L} : \underline{\mathrm{Inn}}(BFV(E)) \to \underline{\mathrm{Hom}}(E,\Pi)$ is a morphism of groups – see Lemma 2.14 and Remark 2.24.

The map L_{geo} was defined as an $\mathrm{Inn}_{\geq 1}(BFV(E))$-invariant extension of L_{nor}, see Corollary 2.19. Since the latter map was surjective – see Theorem 1.13 – so is $\mathcal{L}_{\mathrm{geo}}$.

Surjectivity of $\mathcal{L}_{\mathrm{geo}}$ on the level of morphisms was essentially established in the proof of Theorem 2.25: there a lift $((\hat{\phi} \circ \hat{\psi}) \cdot \delta_0, \hat{\phi} \circ \hat{\psi})$ of a Hamiltonian homotopy $(-\hat{\mu}, \hat{\varphi})$ was constructed such that

$$\hat{L}_{\mathrm{geo}}((\hat{\phi} \circ \hat{\psi}) \cdot \delta_0, \hat{\phi} \circ \hat{\psi}) = (-\hat{\mu}, \hat{\varphi})$$

holds, i.e. $\hat{L}_{\mathrm{geo}}$ is surjective. This implies that $\mathcal{L}_{\mathrm{geo}}$ is surjective on the level of morphisms. $\square$

Definition 3.20. Let $F : \mathcal{G} \to \mathcal{H}$ be a morphism of small groupoids. The *kernel* of F is the subgroupoid $\ker(F)$ of $\mathcal{G}$ with

(a) the set of objects is the set of objects of $\mathcal{G}$,
(b) the set of morphisms from X to Y is the set of all morphisms of $\mathcal{G}$ from X to Y that map to an identity morphism under F.

The *image* of F is the subgroupoid $\mathrm{im}(F)$ of $\mathcal{H}$ with

(a) the set of objects is the set of objects of $\mathcal{H}$ that have nonempty preimage under F,
(b) the set of morphisms from X to Y is the set of all morphisms of $\mathcal{H}$ from X to Y that have nonempty preimage under F.

Lemma 3.21. *The kernel of any morphism of groupoids $F : \mathcal{G} \to \mathcal{H}$ is normal, i.e. let f be a morphism from X to Y in $\mathcal{G}$ and g a morphism from Y to Y in* $\ker(F)$ *then $f^{-1} \circ g \circ f$ is a morphism in* $\ker(F)$.

Proof. We simply compute

$$\begin{aligned} F(f^{-1} \circ g \circ f) &= F(f^{-1}) \circ F(g) \circ F(f) = F(f^{-1}) \circ F(f) \\ &= F(f^{-1} \circ f) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \end{aligned}$$

and consequently $f^{-1} \circ g \circ f$ is in $\ker(F)$. □

Definition 3.22. Let $\mathcal{K}$ be a normal subgroupoid of a small groupoid $\mathcal{G}$.

The *quotient* of $\mathcal{G}/\mathcal{K}$ of $\mathcal{G}$ by $\mathcal{K}$ is the small groupoid with

(a) the set of objects is the set of isomorphism classes of $\mathcal{K}$,
(b) the set of morphisms is the set of equivalence classes of morphisms in $\mathcal{G}$ with respect to the following equivalence relation:

$$\alpha \in \mathrm{Hom}_{\mathcal{G}}(X, Y) \text{ is equivalent to } \beta \in \mathrm{Hom}_{\mathcal{G}}(W, Z) :\Leftrightarrow$$

there are $f \in \mathrm{Hom}_{\mathcal{K}}(X, W)$ and $g \in \mathrm{Hom}_{\mathcal{K}}(Y, Z)$ such that

$$g \circ \alpha = \beta \circ f$$

holds.

There is a natural morphism of groupoids $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{K}$ that is surjective on the set of objects and on all morphism sets.

Remark 3.23. Observe that $\mathcal{K}$ is required to be a *normal* subgroupoid for the quotient $\mathcal{G}/\mathcal{K}$ to be a well-defined groupoid.

Lemma 3.24. *Suppose $F : \mathcal{G} \to \mathcal{H}$ is a morphism of groupoids.*

Then F induces an isomorphism of groupoids

$$[F] : \mathcal{G}/\ker(F) \xrightarrow{\cong} \mathrm{im}(F).$$

Proof. First F restricts to a morphism of groupoids $F|_{\mathrm{im}\,(F)} : \mathcal{G} \to \mathrm{im}\,(F)$ which is surjective on the sets of objects and on all sets of morphisms. The kernel of this restriction coincides with $\ker(F)$ which is a normal subgroupoid of $\mathcal{G}$. Consider the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "F|_{\mathrm{im}\,(F)}"] \arrow[d, "\pi"] & \mathrm{im}\,(F) \\ \mathcal{G}/\ker(F). & \end{tikzcd}$$

Given an object $[X]$ of $\mathcal{G}/\ker(F)$ choose any representative X in $\mathcal{G}$ and set

$$[F]([X]) := F(X).$$

Moreover let $[\alpha]$ be a morphism of $\mathcal{G}/\ker(F)$. Again choose a representative α in $\mathcal{G}$ and set

$$[F]([\alpha]) := F(\alpha)$$

It is straightforward to check that $[F]$ is well-defined and is an isomorphism of groupoids. $\square$

Theorem 3.25. *Let (E, Π) be a coisotropic vector bundle.*
The kernel of the morphism

$$\mathcal{L}_{\mathrm{geo}} : \hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi) \to \hat{\mathcal{C}}(E, \Pi)$$

is the subgroupoid $\hat{\mathcal{D}}^{\geq 1}_{\mathrm{geo}}(E, \Pi)$, see Definition 3.17.

Proof. Suppose $[(\hat{\alpha}, \hat{\phi})]$ represents a pure morphism in $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$, i.e. we can assume without loss of generality that $(\hat{\alpha}, \hat{\phi})$ is pure, hence $\hat{\phi} \in \underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$. Because of

$$\hat{L}_{\mathrm{geo}}(\hat{\alpha}, \hat{\phi}) = (L_{\mathrm{geo}}(\hat{\alpha}), (\mathrm{id})_{t\in[0,1]}) = ((L_{\mathrm{geo}}(\alpha_0))_{t\in[0,1]}, (\mathrm{id})_{t\in[0,1]})$$

the morphism $[(\hat{\alpha}, \hat{\phi})]$ lies in the kernel of $\mathcal{L}_{\mathrm{geo}}$.
On the other hand suppose $[(\hat{\alpha}, \hat{\phi})]$ is a morphism in $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ that lies in the kernel of $\mathcal{L}_{\mathrm{geo}}$. This means that $\hat{L}_{\mathrm{geo}}(\hat{\alpha}, \hat{\phi})$ is $\simeq_H$-equivalent to

$$((L_{\mathrm{geo}}(\alpha_0))_{t\in[0,1]}, (\mathrm{id})_{t\in[0,1]}),$$

i.e. there is an isotopy of Hamiltonian homotopies

$$(-\hat{\mu}, \hat{\varphi})$$

from $(\hat{L}_{\mathrm{geo}}(\hat{\alpha}), \hat{\phi})$ to $((L_{\mathrm{geo}}(\alpha_0))_{t\in[0,1]}, (\mathrm{id})_{t\in[0,1]})$. We want to "lift" this isotopy of Hamiltonian homotopies to an isotopy of gauge homotopies from $(\hat{\alpha}, \hat{\phi})$ to a pure gauge homotopy. Observe that this would imply that the morphism of $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ represented by $(\hat{\alpha}, \hat{\phi})$ is pure.
The first step in the lifting procedure is the construction of a smooth two-parameter family of *normalized* Maurer–Cartan elements $\hat{\beta}$ of (E, Π) such that $L_{\mathrm{nor}}(\hat{\beta}) = -\hat{\mu}$.

Such a family can be constructed in the same fashion that the smooth one-parameter family of normalized Maurer–Cartan elements $\hat{\xi}$ was constructed in the proof of Theorem 2.25. Furthermore the smooth two-parameter family of Hamiltonian diffeomorphisms $\hat{\varphi}$ lifts to a smooth two-parameter family of inner automorphisms $\hat{\psi} := \underline{R}(\hat{\varphi})$. Observe that because $\mu_{(0,s)} = \mu_{(t,1)} = \mu_{(1,s)}$ holds for all $(t,s) \in [0,1]^2$, the same holds for $\hat{\beta}$. And since $\varphi_{(0,s)} = \varphi_{(t,1)} = \mathrm{id}$ is true for arbitrary $(t,s) \in [0,1]^2$, the same is true for $\hat{\psi}$.

The Maurer–Cartan elements α_0 and $\beta_{(0,s)}$ are both geometric and have the same image under L_{geo}. By Corollary 2.19 there is $\eta \in \mathrm{Inn}_{\geq 1}(BFV(E))$ satisfying $\alpha_0 = \eta \cdot \beta_{(0,s)}$. Now consider $\hat{\gamma} := (\hat{\psi}^{-1} \circ \eta) \cdot \hat{\beta}$ or in more detail

$$\gamma_{(t,s)} := (\psi_{(t,s)}^{-1} \circ \eta) \cdot \beta_{(t,s)}.$$

This is a smooth two-parameter family of Maurer–Cartan elements of (E, Π) with $L_{\mathrm{geo}}(\gamma_{(t,s)}) = -\mu_{(0,s)}$. In particular the vanishing locus of the component of $\Omega + \gamma_{(t,s)}$ in $\Gamma(\mathcal{E})$ is $S_{\mu_{(0,s)}}$, i.e. the graph of $\mu_{(0,s)}$. Because $\beta_{(t,s)}$ is a normalized Maurer–Cartan element for all $(t,s) \in [0,1]^2$ the component of $\Omega + \beta_{t,s}$ in $\Gamma(\mathcal{E})$ intersects the zero section of $\mathcal{E} \to E$ transversally. Hence so does the component of $\Omega + \gamma_{(t,s)}$ in $\Gamma(\mathcal{E})$ for all $(t,s) \in [0,1]^2$. We compute

$$\gamma_{(0,s)} = (\psi_{(0,s)}^{-1} \circ \eta) \cdot \beta_{(0,s)} = \eta \cdot \beta_{(0,s)} = \alpha_0,$$

i.e. $\gamma_{(0,s)}$ is constant in $s \in [0,1]$.

Consider the smooth two-parameter family of Maurer–Cartan elements $\eta^{-1} \cdot \hat{\gamma}$ of (E, Π). It satisfies

(i) $\eta^{-1} \cdot \gamma_{(0,s)} = \beta_{(0,s)}$,
(ii) the component of $\eta^{-1} \cdot \gamma_{(t,s)}$ in $\Gamma(\mathcal{E})$ intersects the zero section of $\mathcal{E} \to E$ in $S_{\mu_{(0,s)}}$ for all $(t,s) \in [0,1]^2$ and
(iii) this intersection is transversal for all $(t,s) \in [0,1]^2$.

In the proof of Theorem 2.25 we explained how to apply Proposition 9 in Chapter 6 to the smooth one-parameter family of Maurer–Cartan elements $(\eta^{-1}\gamma_{(t,s)})_{t \in [0,1]}$ – for $s \in [0,1]$ fixed – to obtain a smooth one-parameter family of inner automorphisms $\hat{\vartheta}_s \in \underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$ such that

$$\vartheta_{(t,s)} \cdot (\eta^{-1} \cdot \gamma_{(0,s)}) + BFV_{\geq 1}(E) = \eta^{-1} \cdot \gamma_{(t,s)} + BFV_{\geq 1}(E)$$

holds for all $(t,s) \in [0,1]^2$. Since the construction of $\vartheta_{(t,s)}$ is essentially given by solving an ordinary differential equation with initial value id and generating derivation smoothly depending on $\gamma_{(t,s)}$ the one-parameter family of smooth one-parameter families of inner automorphisms $\hat{\vartheta}_s$ yields a smooth two-parameter family of inner automorphisms $\hat{\theta}$.

Define a smooth two-parameter family of Maurer–Cartan elements $\hat{\chi}$ of (E, Π) by

$$\chi_{(t,s)} := (\psi_{(t,s)} \circ \eta \circ \theta_{(t,s)} \circ \eta^{-1}) \cdot \gamma_{(0,s)} = (\psi_{(t,s)} \circ \eta \circ \theta_{(t,s)} \circ \eta^{-1}) \cdot \alpha_0$$

and calculate

$$\begin{aligned}\chi_{(t,s)} + BFV_{\geq 1}(E) &= (\psi_{(t,s)} \circ \eta \circ \theta_{(t,s)} \circ \eta^{-1}) \cdot \gamma_{(0,s)} + BFV_{\geq 1}(E) \\ &= \psi_{(t,s)} \cdot \gamma_{(t,s)} + BFV_{\geq 1}(E) \\ &= \eta \cdot \beta_{(t,s)} + BFV_{\geq 1}(E)\end{aligned}$$

Since $\hat{\beta}$ is a smooth two-parameter family of *normalized* Maurer–Cartan elements, $\eta \cdot \hat{\beta}$ is a smooth two-parameter family of *geometric* Maurer–Cartan elements and hence so is $\hat{\chi}$. However $(\hat{\chi}, \hat{\psi} \circ \eta \circ \hat{\theta} \circ \eta^{-1})$ is not an isotopy of gauge homotopies because $(\chi_{(1,s)})_{s\in[0,1]}$ is not necessarily constant in $s \in [0,1]$. Observe that

$$\chi_{(1,s)} + BFV_{\geq 1}(E) = \eta \cdot \beta_{(1,s)} + BFV_{\geq 1}(E)$$

and since $(\beta_{(1,s)})_{s\in[0,1]}$ is constant in $s \in [0,1]$, so is the component of $(\chi_{(1,s)})_{s\in[0,1]}$ in $\Gamma(\mathcal{E})$, i.e. $(\chi_{(1,s)})_{s\in[0,1]}$ is a one-parameter family of geometric Maurer–Cartan element such that $(\chi^0_{(1,s)})_{s\in[0,1]}$ is constant. So we obtain a smooth one-parameter family of normalized Maurer–Cartan elements $(\eta^{-1} \circ \chi_{(1,s)})_{s\in[0,1]}$ whose component in $\Gamma(\mathcal{E})$ is constant. Fix $s \in [0,1]$. Following Remark 1.14 and Theorem 2.13, Chapter 4 there is $\delta_{(t,s)} \in \underline{\mathrm{Inn}}_{\geq 2}(BFV(E))$ such that

$$\delta_{(1,s)} \cdot (\eta^{-1} \cdot \chi_{(1,s)}) = \eta^{-1} \cdot \chi_{(1,0)}$$

holds for all $s \in [0,1]$. Theorem 2.13 in Chapter 4 also works for smooth one-parameter families, i.e. $(\delta_{(t,s)})_{t\in[0,1]}$ yield a smooth two-parameter family of automorphisms. Next we define

$$\begin{aligned}&\hat{\omega} := ((\eta \circ \delta_{(t,s)} \circ \eta^{-1} \circ \psi_{(t,s)} \circ \eta \circ \theta_{(t,s)} \circ \eta^{-1}) \cdot \alpha_0) \quad \text{and} \\ &\hat{\zeta} := (\eta \circ \delta_{(t,s)} \circ \eta^{-1} \circ \psi_{(t,s)} \circ \eta \circ \theta_{(t,s)} \circ \eta^{-1}) \quad \text{respectively.}\end{aligned}$$

The pair $(\hat{\omega}, \hat{\zeta})$ is an isotopy of gauge homotopies. Observe that for $s = 1$ we obtain $\zeta_{(t,1)} = \eta \circ \delta_{(t,1)} \circ \theta_{(t,1)} \circ \eta^{-1} \in \underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$, i.e. the gauge homotopy $(\hat{\omega}_{(t,1)}, \hat{\zeta}_{(t,1)})$ is pure.

The isotopy of gauge homotopies $(\hat{\omega}, \hat{\zeta})$ satisfies

$$\hat{L}_{\mathrm{geo}}(\omega_{(t,s)}, \zeta_{(t,s)}) = (-\mu_{(t,s)}, \varphi_{(t,s)})$$

for all $(t,s) \in [0,1]^2$ by construction. In particular the images of $(\omega_{(t,0)}, \zeta_{(t,0)})$ and $(\hat{\alpha}, \hat{\phi})$ under $\hat{L}_{\mathrm{geo}}$ coincide. Consequently $\underline{L}(\zeta_{(t,0)}) = \underline{L}(\phi_t)$ holds for all $t \in [0,1]$, hence $\underline{L}(\phi_t \circ \zeta^{-1}_{(t,0)}) = (\mathrm{id})_{t\in[0,1]}$. Lemma 2.14 implies that there is a unique $\hat{\tau} \in \underline{\mathrm{Inn}}_{\geq 1}(BFV(E))$ such that

$$\phi_t = \tau_t \circ \zeta_{(t,0)}$$

is true for all $t \in [0,1]$. The pair

$$(\tau_t \cdot \omega_{(t,s)}, \tau_t \circ \zeta_{(t,s)})$$

defines an isotopy of gauge homotopies from the gauge homotopy

$$(\tau_t \cdot \omega_{(t,0)}, \tau_t \circ \zeta_{(t,0)}) = ((\tau_t \circ \zeta_{(t,0)}) \cdot \alpha_0, \phi_t) = (\alpha_t, \phi_t)$$

to the gauge homotopy

$$(\tau_t \cdot \omega_{(t,1)}, \tau_t) = ((\tau_t \circ \eta \circ \delta_{(t,1)} \circ \theta_{(t,1)} \circ \eta^{-1}) \cdot \alpha_0, \tau_t \circ \eta \circ \delta_{(t,1)} \circ \theta_{(t,1)} \circ \eta^{-1}).$$

The latter gauge homotopy is pure, hence $(\hat{\alpha}, \hat{\phi})$ is equivalent to a pure gauge homotopy. □

Definition 3.26. Let (E, Π) be a coisotropic vector bundle.

The *BFV-groupoid* $\hat{\mathcal{D}}(E, \Pi)$ of (E, Π) is the quotient of the groupoid geometric Maurer–Cartan elements $\hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi)$ by $\hat{\mathcal{D}}^{\geq 1}_{\mathrm{geo}}(E, \Pi)$.

Corollary 3.27. *Let (E, Π) be a coisotropic vector bundle.*

The morphism of groupoids

$$\mathcal{L}_{\mathrm{geo}} : \hat{\mathcal{D}}_{\mathrm{geo}}(E, \Pi) \to \hat{\mathcal{C}}(E, \Pi)$$

introduced in Lemma 3.19 induces an isomorphism of groupoids $\mathcal{L}$ between the BFV-groupoid $\hat{\mathcal{D}}(E, \Pi)$ of (E, Π) and the groupoid of coisotropic sections $\hat{\mathcal{C}}(E, \Pi)$ of (E, Π).

Proof. By Lemma 3.19 the morphism $\mathcal{L}_{\mathrm{geo}}$ is surjective on the sets of objects and on all morphism sets, hence $\mathrm{im}\,(\mathcal{L}_{\mathrm{geo}}) = \hat{\mathcal{C}}(E, \Pi)$. In Theorem 3.25 the kernel of $\mathcal{L}_{\mathrm{geo}}$ was identified with the subgroupoid $\hat{\mathcal{D}}^{\geq 1}_{\mathrm{geo}}(E, \Pi)$. Consequently $\mathcal{L}_{\mathrm{geo}}$ induces an isomorphism of groupoids

$$\mathcal{L} := [\mathcal{L}_{\mathrm{geo}}] : \hat{\mathcal{D}}(E, \Pi) \xrightarrow{\cong} \hat{\mathcal{C}}(E, \Pi),$$

see Lemma 3.24. □

CHAPTER 6

Technical Remarks

LEMMA 4. *The family of morphisms* $(U_k(t) : \mathcal{S}(\mathfrak{a}) \to \mathfrak{a})_{k\in\mathbb{N}}$ *defined by* $U_0(t) = 0$, $U_1(t) := \Pi_{\mathfrak{a}}\phi_t$ *and*

$$U_k(t)(x_1 \otimes \cdots \otimes x_k) := \sum_{\sigma\in\Sigma_k} \mathrm{sign}(\sigma) \sum_{l\geq 1} \sum_{j_1+\cdots+j_l=k-1} \frac{1}{kl!j_1!\cdots j_l!} \cdots$$
$$\Pi_{\mathfrak{a}}\big([[\cdots[\phi_t(x_{\sigma(1)}), U_{j_1}(t)(x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(j_1+1)})], \cdots$$
$$\cdots], U_{j_l}(t)(x_{\sigma(j_1+\cdots+j_{(l-1)}+2)} \otimes \cdots \otimes x_{\sigma(k)})]\big).$$

which was introduced in the proof of Theorem 3.7 in Chapter 2 satisfies the family of ordinary differential equations

$$\frac{d}{dt}|_{t=s} U_k(t)(x_1 \otimes \cdots \otimes x_k)$$
$$= \sum_{l\geq 1} \sum_{j_1+\cdots+j_l=k} \sum_{\tau\in\Sigma_k} \frac{\mathrm{sign}(\tau)}{l!j_1!\cdots j_l!} D_l^{X_s}(U_{j_1}(s)(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma_{j_1}}) \otimes \cdots$$
$$\cdots \otimes U_{j_l}(s)(x_{\sigma(j_1+\cdots+j_{(l-1)}+1))} \otimes \cdots \otimes x_{\sigma(k)}))$$

and the initial conditions $U_1(0) = \mathrm{id}$ *and* $U^k(0) = 0$ *for* $k \neq 1$.

PROOF. By definition $U_1(t) = \Pi_{\mathfrak{a}}\phi_t$ so $U_1(0) = \mathrm{id}$ and

$$\frac{d}{dt}|_{t=s} U_1(t)(\cdot) = \Pi_{\mathfrak{a}}([X_s, \phi_s(\cdot)]) = \Pi_{\mathfrak{a}}([X_s, \Pi_{\mathfrak{a}}\phi_s(\cdot)]) = (D_1^{X_s} \circ U_1(s))(\cdot)$$

is satisfied.

Suppose we established the ordinary differential equation for $(U_l(t))_{l<k}$. We want to prove that this implies that the ordinary differential equation is satisfied for $U_k(t)$ too. The definition of $U_k(t)$ implies

$$\frac{d}{dt}|_{t=s} U_k(t)(x_1 \otimes \cdots \otimes x_k) = \sum_{\sigma\in\Sigma_k} \mathrm{sign}(\sigma) \sum_{l\geq 1} \sum_{j_1+\cdots+j_l=k-1} \frac{1}{kl!j_1!\cdots j_l!} \cdots$$
$$\Bigg(\Pi_{\mathfrak{a}}\big([[\cdots[[[X_s, \phi_s], U_{j_1}], U_{j_2}], \cdots], U_{j_l}]\big)$$
$$+ l\Pi_{\mathfrak{a}}\big([[[\cdots[\phi_s, U_{j_1}], \cdots], U_{j_{(l-1)}}], \frac{d}{dt}|_{t=s} U_{j_l}(t)]\big)\Bigg)$$

where we suppressed the arguments $(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ and $s \in [0, 1]$. The first term comes from deriving ϕ_t, the second one from deriving one of the factors $U_m(t)$ with $m < k$. We denote the two terms by A_k and B_k respectively.

$A_k(s)$ contains terms of the form $[\cdots[[[X_s,\phi_s],U_{j_1}(s)],U_{j_2}(s)],\cdots]$. Using the graded Jacobi identity consecutively yields

$$A_k = \sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq 1}\sum_{r+s=l}\sum_{\substack{\alpha_1+\cdots+\alpha_r+\\+\beta_1+\cdots+\beta_s=k-1}}\frac{1}{kr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}\cdots$$
$$\Pi_{\mathfrak{a}}\Big([\big([[\cdots[[X_s,U_{\alpha_1}],U_{\alpha_2}],\cdots],U_{\alpha_r}]\big),\big([[\cdots[[\phi_s,U_{\beta_1}],U_{\beta_2}],\cdots],U_{\beta_s}]\big)\Big).$$

Applying $\Pi_{\mathfrak{a}}[\cdot,\cdot]=\Pi_{\mathfrak{a}}[\Pi_{\mathfrak{a}}(\cdot),\cdot]+\Pi_{\mathfrak{a}}[\cdot,\Pi_{\mathfrak{a}}(\cdot)]$ leads to

$$\begin{aligned}
A_k(s) &= \\
&= \Bigg(\sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq 1}\sum_{r+s=l}\sum_{\substack{\alpha_1+\cdots+\alpha_r+\\+\beta_1+\cdots+\beta_s=k-1}}\frac{1}{kr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}\cdots \\
&\qquad D^{X_s}_{(r+1)}\big(U_{\alpha_1}\otimes\cdots\otimes U_{\alpha_r}\otimes\Pi_{\mathfrak{a}}([[\cdots,[\phi_s,U_{\beta_1}]\cdots],U_{\beta_s}])\big)\Bigg) \\
&- \Bigg(\sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq 1}\sum_{r+s=l}\sum_{\substack{\alpha_1+\cdots+\alpha_r+\\+\beta_1+\cdots+\beta_s=k-1}}\frac{1}{kr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}\cdots \\
&\qquad \Pi_{\mathfrak{a}}[\big([[\cdots[\phi_s,U_{\alpha_1}],\cdots],U_{\alpha_r}]\big),\Pi_{\mathfrak{a}}\big([[\cdots[[X_s,U_{\beta_1}],U_{\beta_2}],\cdots],U_{\beta_s}]\big)]\Bigg).
\end{aligned}$$

We claim that the identity

$$\begin{aligned}
&\sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq 1}\sum_{j_1+\cdots+j_l=k-1}\frac{1}{k(l-1)!j_1!\cdots j_l!}\cdots \\
&\qquad \Pi_{\mathfrak{a}}\big([[[\cdots[\phi_s,U_{j_1}],\cdots],U_{j_{(l-1)}}],\frac{d}{dt}|_{t=s}U_{j_l}(t)]\big) \\
&= \sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq 1}\sum_{r+s=l}\sum_{\substack{\alpha_1+\cdots+\alpha_r+\\+\beta_1+\cdots+\beta_s=k-1}}\frac{1}{kr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}\cdots \\
&\qquad \Pi_{\mathfrak{a}}[\big([[\cdots[\phi_s,U_{\alpha_1}],\cdots],U_{\alpha_r}]\big),\Pi_{\mathfrak{a}}\big([[\cdots[[X_s,U_{\beta_1}],U_{\beta_2}],\cdots],U_{\beta_s}]\big)]
\end{aligned}$$

is true. This means that $B_k(s)$ cancels the second term in the expression for $A_k(s)$ given above. The identity is derived easily by applying the induction hypothesis about $\frac{d}{dt}|_{t=s}U_m(t)$ for $m<k$ to $B_k(s)$.

Finally we claim that the equality

$$\begin{aligned}
&\sum_{\sigma\in\Sigma_k}\mathrm{sign}(\sigma)\sum_{l\geq 1}\sum_{r+s=l}\sum_{\substack{\alpha_1+\cdots+\alpha_r+\\+\beta_1+\cdots+\beta_s=k-1}}\frac{1}{kr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}\cdots \\
&\qquad D^{X_s}_{(r+1)}\big(U_{\alpha_1}\otimes\cdots\otimes U_{\alpha_r}\otimes\Pi_{\mathfrak{a}}([[\cdots,[\phi_s,U_{\beta_1}]\cdots],U_{\beta_s}])\big) \\
&= \sum_{l\geq 1}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{\mathrm{sign}(\tau)}{l!j_1!\cdots j_l!}D^{X_s}_l(U_{j_1}\otimes\cdots\otimes U_{j_l})
\end{aligned}$$

holds which implies that $U_k(t)$ satisfies the ordinary differential equation. To establish the equality we first use the recursive definition of $U_m(t)$ to arrange the terms of the form $\Pi_{\mathfrak{a}}[[\cdots[\phi_t, U_{\beta_1}(t)],\cdots], U_{\beta_s}(t)]$ into some $U_\beta(t)$. One arrives at

$$\sum_{l\geq 1}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{\mathrm{sign}(\tau)}{k(l-1)!(j_1-1)!j_2!\cdots j_l!}D_l^{X_s}(U_{j_1}\otimes\cdots\otimes U_{j_l}).$$

It remains to prove that this is equal to

$$\sum_{l\geq 1}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{\mathrm{sign}(\tau)}{l!j_1!\cdots j_l!}D_l^{X_s}(U_{j_1}\otimes\cdots\otimes U_{j_l}).$$

We construct a third map for which it is easy to show that is it equal to the last two expressions, hence they coincide.

Assume one is given k distinguishable objects and l boxes. Each of the box can be filled with a fixed number of objects which is bigger than 0. The number of boxes that can contain exactly j_i objects is w_i ($0 < j_1 < \cdots < j_m$). The identities $w_1 + \cdots + w_m = r$ and $w_1 j_1 + \cdots w_m j_m = k$ must be satisfied. We label this situation by $(l|(j_1, w_1), \ldots, (j_l, w_l))$ and denote the number $w_1 j_1 + \cdots w_m j_m$ by $|(l|(j_1, w_1), \ldots, (j_l, w_l))|$. Furthermore we assume that boxes that can be filled by the same number of objects are indistinguishable. The number of different ways to put the k objects into these boxes is given by

$$\frac{k!}{w_1!\cdots w_m!(j_1!)^{w_1}\cdots(j_m!)^{w_m}}.$$

Consider

$$\sum_{l\geq 1}\sum_{|(l|(j_1,w_1),\ldots,(j_m,w_m))|=k}\sum_{\tau\in\Sigma_k}\frac{\mathrm{sign}(\tau)}{w_1!\cdots w_m!(j_1!)^{w_1}\cdots(j_m!)^{w_m}}\cdots$$
$$D_l^{X_s}(U_{j_1}\otimes\cdots\otimes U_{j_1}\otimes\cdots\otimes U_{j_m}\otimes\cdots\otimes U_{j_m}).$$

It is straightforward to check that this map is equal to

$$\sum_{l\geq 1}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{\mathrm{sign}(\tau)}{k(l-1)!(j_1-1)!j_2!\cdots j_l!}D_l^{X_s}(U_{j_1}\otimes\cdots\otimes U_{j_l})$$

on the one hand and to

$$\sum_{l\geq 1}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k}\frac{\mathrm{sign}(\tau)}{l!j_1!\cdots j_l!}D_l^{X_s}(U_{j_1}\otimes\cdots\otimes U_{j_l})$$

on the other hand. □

LEMMA 5. *Given a Poisson manifold (M,Π), the composition of diffeomorphisms equips the sets $\underline{\mathrm{Ham}}(E,\Pi)$ and $\mathrm{Ham}(M,\Pi)$ with the structure of groups.*

PROOF. Let F and G be two smooth functions on $M \times [0,1]$ and $(\varphi_t)_{t\in[0,1]}$ and $(\phi_t)_{t\in[0,1]}$ the two one-parameter families of Hamiltonian diffeomorphisms generated by F and G respectively, i.e. $\varphi_0 = \mathrm{id} = \phi_0$ and

$$\frac{d}{dt}|_{t=s}\varphi_t = X_{F_s}|_{\varphi_s}, \quad \frac{d}{dt}|_{t=s}\phi_t = X_{G_s}|_{\varphi_s}$$

hold for arbitrary $s \in [0,1]$. We compute

$$\begin{aligned} \frac{d}{dt}|_{t=s}(\phi_t \circ \varphi_t)^*(\cdot) &= \frac{d}{dt}|_{t=s}(\varphi_t^* \circ \phi_t^*)(\cdot) \\ &= \varphi_s^*([X_{F_s}, (\phi_s)^*(\cdot)]_{SN}) + (\varphi_s^* \circ \phi_s^*)([X_{G_s}, \cdot]_{SN}) \\ &= (\varphi_s^* \circ \phi_s^*)([(\phi_s^{-1})^* X_{F_s}, \cdot]_{SN}) + (\varphi_s^* \circ \phi_s^*)([X_{G_s}, \cdot]_{SN}) \\ &= (\phi_s \circ \varphi_s)^* \left([(\phi_s^{-1})^* X_{F_s} + X_{G_s}, \cdot]_{SN}\right). \end{aligned}$$

Hence the smooth one-parameter family of diffeomorphisms $(\phi_t \circ \varphi_t)_{t\in[0,1]}$ is generated by the smooth one-parameter family of vector fields

$$((\phi_t^{-1})^* X_{F_t} + X_{G_t})_{t\in[0,1]}.$$

Next we calculate

$$\begin{aligned} (\phi_t^{-1})^* X_{F_t} &= (\phi_t^{-1})^*(-[\Pi, F_t]_{SN}) = -[(\phi_t^{-1})^*(\Pi), (\phi_t^{-1})^*(F_t)]_{SN} \\ &= -[\Pi, F_t \circ \phi_t^{-1}]_{SN} = X_{F_t\circ\phi_t^{-1}}. \end{aligned}$$

We applied Corollary 1.14 in Chapter 3 to obtain the last equality. Consequently

$$(\phi_t^{-1})^* X_{F_t} + X_{G_t} = X_{F_t\circ\phi_t^{-1}} + X_{G_t} = X_{\left(F_t\circ\phi_t^{-1}+G_t\right)},$$

i.e. $(\phi_t \circ \varphi_t)_{t\in[0,1]}$ is a smooth one-parameter family of diffeomorphisms starting at the identity and generated by the smooth one-parameter family of Hamiltonian vector fields associated to the function

$$H : M \times [0,1] \to \mathbb{R}, \quad H(x,t) := F(\phi_t^{-1}(x), t) + G(x,t).$$

Similarly one checks that $(\phi_t^{-1})_{t\in[0,1]}$ is a smooth one-parameter family of diffeomorphisms starting at the identity and generated by the smooth one-parameter family of Hamiltonian vector fields associated to the function

$$K : M \times [0,1] \to \mathbb{R}, \quad K(x,t) := -G(\phi_t^{-1}(x), t).$$

These two calculations imply that $\underline{\mathrm{Ham}}(M,\Pi)$ is a group with respect to composition. That $\mathrm{Ham}(M,\Pi)$ is also a group with respect to composition is an immediate consequence. □

LEMMA 6. *Let A and B be two manifold of the same dimension, K a compact locally metrizable topological space and Θ a continous map from $A \times K$ to B such that*

(a) *the restriction Θ_u of Θ to $A \times \{u\}$ is an embedding of A into B for arbitrary $k \in K$ and*

(b) *there are submanifolds X and Y of A and B respectively and a fixed diffeomorphism $\phi : X \xrightarrow{\cong} Y$ such that the restriction of Θ to $X \times \{u\}$ is equal to ϕ for every $u \in K$*

hold. Then the intersection $\bigcap_{u \in k} \Theta_u(A)$ contains an open neighbourhood of Y in B.

Proof. Observe that this is a local claim: the intersection

$$C := \bigcap_{u \in K} \Theta_u(A)$$

is an open neighbourhood of Y in B if it is true for every point $y \in Y$.

Fix an arbitrary $y \in Y$. We have to show that there is an open neighbourhood U_y of y in B that is contained in C. We first prove that for every $u \in K$ there exists an open neighbourhood $V_y(u)$ of y in B which is contained in C and $\delta(u) > 0$ such that for all $v \in K$ with distance less than $\delta(u)$ from u

$$V_y(u) \subset \Theta_v(A)$$

holds.

Because Θ_u is an embedding and the dimensions of A and B match, the set $\Theta_u(A)$ is an open neighbourhood of y in B, i.e. for any chart of B centered at y there is a $\epsilon(y) > 0$ such that the open ball $B_{3\epsilon(y)}(y)$ centered at y with radius $3\epsilon(y)$ is contained in $\Theta_u(A)$. Consider the sphere $S_{2\epsilon(y)}(y)$ centered at y with radius $2\epsilon(y)$. Pick a point z on this sphere. Since Θ is continuous the preimage of $B_{\epsilon(y)}(z)$ is an open subset of $A \times K$. Denote the preimage of z under Θ_u by $a \in A$. Obviously (a, u) is contained in the preimage of $B_{\epsilon(y)}(z)$and since this preimage is open we can find an open neighbourhood of (a, u) in $A \times K$ which is contained in this preimage. Since both A and K are locally metrizable, so is the product and consequently we can find $d(z) > 0$ and $\rho(z) > 0$ such that the product $B_{d(z)}(a) \times B_{\rho(z)}(u)$ lies in the preimage of $B_{\epsilon(y)}(z)$ under Θ, i.e. the image of $B_{d(z)}(a) \times B_{\rho(z)}$ under Θ lies in $B_{\epsilon(y)}(z)$.

The family of subsets $(B_{d(z)}(\Theta_u^{-1}(z)))_{z \in S_{2\epsilon(y)}(y)}$ is an open cover of the compact set $\Theta_u^{-1}(S_{2\epsilon(y)}(y)) \subset A$. Hence we can find a finite subcover given by $(B_{d(z_i)}(a_i))_{i=1,\dots,m}$ with $a_i := \Theta_u^{-1}(z)$. Now set

$$\rho := \min\left(\rho(z_1), \dots, \rho(z_m)\right) > 0$$

and observe that for every $x \in \Theta_u^{-1}(S_{2\epsilon(y)}(y))$ and $v \in K$ with distance less than ρ from u the image $\Theta_v(x)$ is contained in

$$\{x \in B : \epsilon(y) < |x - y| < 3\epsilon(y)\}.$$

This implies that the image of $\Theta_u^{-1}(B_{3\epsilon(y)}(y)) \times B_\rho(u)$ contains

$$\{x \in B : |x - y| < \epsilon(y)\}.$$

In more detail $\Theta_u^{-1}(B_{3\epsilon(y)}(y))$ is contractible and hence its image under Θ_v is contractible too. Moreover we proved that for $v \in K$ with distance less than ρ from u the image of $\Theta_u^{-1}(B_{3\epsilon(y)}(y))$ under Θ_v contains a manifold diffeomorphic to the sphere which is contained in

$$\{x \in B : \epsilon(y) < |x - y| < 3\epsilon(y)\}.$$

Because y also lies in the contractible image of $\Theta_u^{-1}(B_{3\epsilon(y)}(y))$ under Θ_v, the image contains $B_{\epsilon(y)}(y)$ for arbitrary $v \in B_\rho(u)$. Consequently the image of A under Θ_v contains $B_{\epsilon(y)}(y)$ too for all $v \in B_\rho(u)$.

We proved that for every $u \in K$ there is an open neigbourhood $V_y(u) := B_{\epsilon(y)}(y)$ of y in B and an open neighbourhood $B_\rho(u)$ of u in K such that

$$V_y(u) \subset \Theta_v(A)$$

holds for all $v \in B_\rho(u)$. Notice that we suppressed the dependence of ρ on u.

The family of sets $(B_\rho(u))_{u\in K}$ is an open cover of K. Because K is compact we can find a finite subcover $(B_{\rho_i}(u_i))_{i=1,\ldots,n}$ and we set

$$U_y := \bigcap_{i=1,\ldots,n} V_y(u_i).$$

This is an open subset of B satisfying $y \in U_y$ and by construction it is contained in

$$C = \bigcap_{u\in K} \Theta_u(A).$$

$\square$

Lemma 7. *The linear operator h on $BFV(E)$ of degree -1 introduced in the proof of Lemma 2.7 in Chapter 4 satisfies the relation*

$$[h, \delta] = h \circ \delta + \delta \circ h = \mathrm{id} - (\wedge p^*) \circ (\wedge i^*).$$

Proof. Consider the following coordinates on $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$: $(x^\beta)_{\beta=1,\ldots,s}$ are local coordinates on S, $(y^i)_{i=1,\ldots,e}$ are linear coordinates along the fibres of E, $(b^i)_{i=1,\ldots,e}$ is the corresponding local frame on $\mathcal{E}^*$ and $(c_i)_{i=1,\ldots,e}$ the dual frame on $\mathcal{E}$. These two frames yield a frame of $\wedge(\mathcal{E} \oplus \mathcal{E}^*)$. Locally the differential δ is given by

$$\sum_{i=1}^{e} y^i \frac{\partial}{\partial b^i}.$$

and h is given by

$$h(f(x, y, c)b^{i_1} \cdots b^{i_k}) := \sum_{j=1}^{e} b^j \left(\int_0^1 \frac{\partial f}{\partial y^j}(x, t \cdot y, c)t^k \right) b^{i_1} \cdots b^{i_k}.$$

We compute $(\mathrm{id}-(\wedge p^*)\circ(\wedge i^*))$ first:

$$\begin{aligned}
&(\mathrm{id}-(\wedge p^*)\circ(\wedge i^*))(f(x,y,c)b^{i_1}\cdots b^{i_k})\\
&\quad= f(x,y,c)b^{i_1}\cdots b^{i_k} - f(x,0,c)|_{b^j\equiv 0}\\
&\quad= \int_0^1 \frac{d}{dt}\left(f(x,t\cdot y,c)(tb^{i_1})\cdots(tb^{i_k})\right)dt\\
&\quad= \int_0^1 \frac{d}{dt}\left(f(x,t\cdot y,c)t^k\right)dt b^{i_1}\cdots b^{i_k}\\
&\quad= \int_0^1 \sum_{j=1}^{e}\left(\frac{\partial f}{\partial y^j}(x,t\cdot y,c)\cdot y^j\right)t^k dt(b^{i_1}\cdots b^{i_k})\\
&\qquad\qquad +k\left(\int_0^1 f(x,t\cdot y,c)t^{(k-1)}dt\right)(b^{i_1}\cdots b^{i_k}).
\end{aligned}$$

Next we calculate

$$\begin{aligned}
&(h\delta)(f(x,y,c)b^{i_1}\cdots b^{i_k}) = h\left(\sum_{\alpha=1}^{k}(-1)^{(\alpha-1)}f(x,y,c)y^{i_\alpha}(b^{i_1}\cdots\hat{b}^{i_\alpha}\cdots b^{i_k})\right)\\
&= \sum_{j=1}^{e}\sum_{\alpha=1}^{k}(-1)^{(\alpha-1)}b^j\left(\int_0^1\frac{\partial(fy^{i_\alpha})}{\partial y^j}(x,t\cdot y,c)t^{(k-1)}dt\right)(b^{i_1}\cdots\hat{b}^{i_\alpha}\cdots b^{i_k})\\
&= \sum_{j=1}^{e}\sum_{\alpha=1}^{k}(-1)^{(\alpha-1)}\left(\int_0^1\frac{\partial f}{\partial y^j}(x,t\cdot y,c)y^{i_\alpha}t^k dt\right)(b^{i_1}\cdots\hat{b}^{i_\alpha}\cdots b^{i_k})\\
&\quad +\sum_{\alpha=1}^{k}\left(\int_0^1 f(x,t\cdot y,c)t^{(k-1)}dt\right)(b^{i_1}\cdots b^{i_k})\\
&= \sum_{j=1}^{e}\sum_{\alpha=1}^{k}(-1)^{(\alpha-1)}\left(\int_0^1\frac{\partial f}{\partial y^j}(x,t\cdot y,c)y^{i_\alpha}t^k dt\right)(b^{i_1}\cdots\hat{b}^{i_\alpha}\cdots b^{i_k})\\
&\quad +k\left(\int_0^1 f(x,t\cdot y,c)t^{(k-1)}dt\right)(b^{i_1}\cdots b^{i_k})
\end{aligned}$$

and

$$\begin{aligned}
&(\delta h)(f(x,y,c)b^{i_1}\cdots b^{i_k}) =\\
&= \delta\left(\sum_{j=1}^{e}b^j\left(\int_0^1\frac{\partial f}{\partial y^j}(x,t\cdot y,c)t^k dt\right)b^{i_1}\cdots b^{i_k}\right)\\
&= \sum_{j=1}^{e}y^j\left(\int_0^1\frac{\partial f}{\partial y^j}(x,t\cdot y,c)t^k dt\right)(b^{i_1}\cdots b^{i_k})\\
&\quad +\sum_{j=1}^{e}b^j\left(\int_0^1\frac{\partial f}{\partial y^j}(x,t\cdot y,c)t^k dt\right)\left(\sum_{\alpha=1}^{k}(-1)^{\alpha}y^{i_\alpha}b^{i_1}\cdots\hat{b}^{i_\alpha}\cdots b^{i_k}\right).
\end{aligned}$$

We conclude with

$$
\begin{aligned}
& (h\delta + \delta h)(f(x,y,c)b^{i_1}\cdots b^{i_k}) = \\
= \quad & \int_0^1 \sum_{j=1}^{e} \left(\frac{\partial f}{\partial y^j}(x, t\cdot y, c)\cdot y^j \right) t^k dt (b^{i_1}\cdots b^{i_k}) \\
& +k \left(\int_0^1 f(x, t\cdot y, c) t^{(k-1)} dt \right) (b^{i_1}\cdots b^{i_k}) \\
= \quad & (\mathrm{id} - (\wedge p^*)\circ(\wedge i^*))(f(x,y,c)b^{i_1}\cdots b^{i_k}).
\end{aligned}
$$

□

LEMMA 8. *The vanishing ideal of $\mathbb{R}^k \times \{0\}$ in $\mathbb{R}^{(k+e)} \cong \mathbb{R}^k \oplus \mathbb{R}^e$ is generated by the linear fibre coordinates on $\mathbb{R}^e$.*

PROOF. That the ideal generated by the linear fibre coordinates on $\mathbb{R}^e$ is contained in the vanishing ideal of $\mathbb{R}^k \times \{0\}$ in $\mathbb{R}^{(k+e)}$ is evident.

The reverse inclusion is proved as follows: Let h be a smooth function on $\mathbb{R}^{(k+e)}$ whose restriction to $\mathbb{R}^k \times \{0\}$ vanishes. Then

$$
\begin{aligned}
h(x,y) \quad &= \quad h(x,y) - h(x,0) = \int_0^1 \frac{d}{dt}\left(h(x, t\cdot y)\right) dt \\
&= \quad \int_0^1 \left(\sum_{i=1}^{e} \frac{\partial h}{\partial y^i}(x, t\cdot y) y^i \right) dt = \sum_{i=1}^{e} y^i \left(\int_0^1 \frac{\partial h}{\partial y^i}(x, t\cdot y) dt \right)
\end{aligned}
$$

holds, i.e. $h(x,y)$ lies in the ideal generated by the linear fibre coordinates on $\mathbb{R}^e$. □

PROPOSITION 9. *Let $E \to S$ be a vector bundle and consider the pull back $\mathcal{E} \to E$ of $E \to S$ along $E \to S$. Suppose β_t is a smooth one-parameter family of sections of $\mathcal{E} \to E$, i.e. a section of the pull back $\mathcal{E}_{[0,1]}$ of $\mathcal{E} \to E$ along $E \times [0,1] \to E$, satisfying the following conditions:*

(a) *The section β_0 is equal to the tautological section Ω_0 of $\mathcal{E} \to E$*
(b) *The intersection of the graph of β_t with the zero section of $\mathcal{E} \to E$ is S.*
(c) *This intersection is transversal, i.e.*

$$T_x(\mathrm{graph}(\beta_t)) + T_x E = T_x \mathcal{E}$$

holds for all $x \in S$.

Then there is a smooth one-parameter family of sections A_t of $GL_+(\mathcal{E})$ such that

$$A_t \cdot \Omega_0 = \Omega_t$$

is true for all $t \in [0,1]$.

Moreover the statement remains true for smooth one-parameter families of sections of the restriction of $\mathcal{E} \to E$ to an open neighbourhood U of S in E that is contractible along fibres, see Definition 2.24 in Chapter4.

Proof. We apply the homotopy h introduced in the proof of Lemma 2.7 in Chapter 4 to the smooth-one parameter family $(\beta_t)_{t\in[0,1]}$ and obtain a smooth one-parameter family

$$M_t := -h(\beta_t)$$

of elements of $\Gamma(\mathcal{E}\otimes\mathcal{E}^*)\cong\Gamma(\mathrm{End}(\mathcal{E}))$. The identity $\delta(\beta_t)=[\Omega_0,\beta_t]_G=0$ and

$$[h,\delta]=h\circ\delta+\delta\circ h=\mathrm{id}-(\wedge p^*)\circ(\wedge i^*)$$

which was established in Lemma 7 imply

$$M_t\cdot\Omega_0=[\Omega_0,M_t]_G=\delta(M_t)=\beta_t-(\wedge p^*)\circ(\wedge i^*)(\beta_t).$$

Moreover $(\wedge i^*)(\beta_t)=0$ because part of the definition of $\wedge i^*$ is restriction to S and β_t vanishes there for all $t\in[0,1]$. Hence

$$M_t\cdot\Omega_0=\beta_t$$

holds for all $t\in[0,1]$.

Let ξ be a section of $\mathcal{E}$. In local coordinates one checks that

$$M_t|_S\cdot\xi=-h(\beta_t)|_S\cdot\xi=-\xi\cdot\beta_t$$

where $\xi\cdot$ is the fibre derivative along ξ, see Remark 3.10 in Chapter 3. That the fibrewise derivative of $-\beta_t$

$$E_x\longrightarrow E_x,\quad \xi\mapsto(\xi\cdot(-\beta_t))|_x$$

is fibrewise invertible at every point $x\in S$ is equivalent to the statement that the intersection of $\mathrm{graph}(\beta_t)$ and the zero section of $\mathcal{E}\to E$ is transversal at S. One the other hand the former statement is equivalent to

$$M_t|_S$$

being a fibrewise invertible endomorphism. Consequently the restriction of $(M_t)_{t\in[0,1]}$ to S is a smooth one-parameter family of sections of $GL(\mathcal{E}|_S)\cong GL(E)$.

The map

$$\det(m):E\times[0,1]\xrightarrow{M_t}\mathcal{E}\otimes\mathcal{E}^*\xrightarrow{\det}\wedge^{\mathrm{top}}(\mathcal{E}\otimes\mathcal{E}^*)\cong\mathbb{R}\times E$$

is smooth. Denote the projections from $E\times[0,1]$ to E and $[0,1]$ by pr_1 and pr_2 respectively. For every $(y,t)\in S\times[0,1]$ we have $\det(m)(y,t)\neq 0$, hence there is an open neighbourhoods $V(y,t)$ of (y,t) in $E\times[0,1]$ such that for all $(z,s)\in V(y,t)$ the endomorphism $M_s(z)$ is invertible. This yields an open cover $\mathrm{pr}_2(V(y,t))$ of $[0,1]$ and by compactness there is a finite subcover $(\mathrm{pr}_2(V(y,t_1)),\dots,\mathrm{pr}_2(V(y,t_N)))$. Hence

$$V(y):=\bigcap_{i=1,\dots,N}\mathrm{pr}_1(V(y,t_i))$$

is an open neighbourhood of y in E such that for all $z \in V(y)$ the endomorphism $M_t(z)$ in invertible for arbitrary $t \in [0,1]$. Taking the union

$$V := \bigcup_{y \in S} V(y)$$

yields an open neighbourhood of S in E where M_t is invertible for all $t \in [0,1]$.

Define a smooth one-parameter family of endomorphisms of $\mathcal{E}|_V$ by

$$m_t := \left(\frac{d}{dt} M_t\right) M_t^{-1}.$$

It satisfies

$$m_t(\beta_t) = \left(\frac{d}{dt} M_t\right)(\Omega^0)$$

on V and for arbitrary $t \in [0,1]$. Choose a fibre metric g on $\mathcal{E} \to E$ and define a smooth one-parameter family of fibre wise linear endomorphisms n_t of $\mathcal{E}$ on the complement of S by

$$n_t(z) : \mathcal{E}_z \xrightarrow{P(g)} < (\beta_t)|_z > \to < \left(\frac{d}{dt} M_t\right)(\Omega^0)|_z > \hookrightarrow \mathcal{E}_z.$$

Here $P(g)$ denotes the orthogonal projection of $\mathcal{E}_z$ to $< (\beta_t)_z >$ with respect to g. By definition this smooth one-parameter family $(n_t)_{t \in [0,1]}$ of sections of $\mathcal{E}|_{E \setminus S}$ satisfies

$$n_t(\beta_t) = \left(\frac{d}{dt} M_t\right)(\Omega^0)$$

on $E \setminus S$ and for arbitrary $t \in [0,1]$.

It is possible to find an open neighbourhood W of S in E such that its closure $\overline{W}$ is still contained in V. In fact, choose an embedding of the normal bundle of S in V and equip it with a fibre metric. Let W be given by all elements of the normal bundle with distance less than 1 from the base point over which they lie. Then the closure of W is given by all the elements of the normal bundle with distance less than or equal to 1 from the base point over which they lie. This is still contained in V. Consequently $(V, E \setminus \overline{W})$ is an open cover of E and hence there is a partition of unity (ρ_1, ρ_2) of E subordinated to this open cover, see [**Mi**] for instance. We set

$$a_t := \rho_1 m_t + \rho_2 n_t$$

which is a smooth one-parameter family of sections of $\mathcal{E}$ defined over all of E. It satisfies

$$a_t(\beta_t) = \left(\frac{d}{dt} M_t\right)(\Omega^0)$$

for arbitrary $t \in [0,1]$.

The ordinary differential equation

$$\frac{d}{dt}A_t = a_t \circ A_t, \quad A_0 = \mathrm{id}$$

can be solved fibre wise and one obtains a smooth one-parameter family of sections $(A_t)_{t\in[0,1]}$ of $GL_+(\mathcal{E})$. Furthermore one verifies easily

$$\frac{d}{dt}(A_t(\Omega^0)) = a_t(A_t(\Omega^0)), \quad A_0(\Omega^0) = \Omega^0$$

which is exactly the flow equation satisfied by $(\beta_t)_{t\in[0,1]}$ since

$$\frac{d}{dt}(\beta_t) = \frac{d}{dt}(M_t(\Omega^0)) = \left(\frac{d}{dt}M_t\right)(\Omega^0) = a_t(\beta_t), \quad \beta_0 = \Omega^0.$$

This implies that the equality

$$A_t \cdot \Omega^0 = \beta_t$$

holds for all $t \in [0,1]$.

Since the homotopy h can be restricted to open neighbourhoods of S in E that are contractible along fibres, the whole procedure can be repeated for smooth one-parameter families not defined over all of E but over any open neighbourhood of S in E which is contractible along fibres. $\square$

LEMMA 10. *The signs of the evaluation of the two maps L_n and R_n introduced in the proof of Theorem 3.6 in Chapter 4 on homogeneous elements $\xi_1,\ldots,\xi_n$ of $\Gamma(\wedge E)$ are*

$$\begin{aligned} l_n(\xi_1,\ldots,\xi_n) &= n + \sum_{i=1}^{n-2}(n-i+1)(|\xi_1|+1) \quad \textit{and} \\ r_n(\xi_1,\ldots,\xi_n) &= (n-1) + \sum_{i=1}^{n-2}(n-i+1)(|\xi_1|+1) \end{aligned}$$

respectively.

PROOF. The homological transfer of L_∞-algebra structures described in Section 2, Chapter 2 uses an $L_\infty[1]$-algebra structure as its input data. Hence we first have to transfer the differential graded Lie algebra

$$(BFV(E), [\Omega,\cdot]_{BFV}, [\cdot,\cdot]_{BFV})$$

to this setting with the help of the décalage-isomorphism. In this process the bracket $[\xi_i,\xi_j]_{BFV}$ picks up the sign $(-1)^{(|\xi_i|+1)}$. The structure map L_n involves $(n-1)$ such factors and hence we pick up the sign of (-1) to

$$\sum_{i=1}^{n-1}(|\xi_i|+1).$$

Moreover every homotopy comes with a minus sign. The maps L_n and R_n contain $(n-2)$ and $(n-1)$ copies of h respectively. Finally we transfer the induced structure maps back to the skew-symmetric setting where we pick up the sign of (-1) to

$$\sum_{i=1}^{n}(n-i)(|\xi_i|+1).$$

Summing up all the exponents yields

$$\begin{aligned}
&\sum_{i=1}^{n-1}(|\xi_i|+1)+(n-2)+\sum_{i=1}^{n}(n-i)(|\xi_i|+1)+\mathbb{Z}_2 = \\
&= \quad n+\sum_{i=1}^{n-1}(n-i+1)(|\xi_i|+1)+\mathbb{Z}_2 \\
&= \quad n+\sum_{i=1}^{n-2}(n-i+1)(|\xi_i|+1)+\mathbb{Z}_2
\end{aligned}$$

for L_n and similarly

$$(n-1)+\sum_{i=1}^{n-2}(n-i+1)(|\xi_1|+1)+\mathbb{Z}_2$$

for R_n. □

Lemma 11. *Let $[\cdot,\cdot]_G$ be the graded Poisson bracket of degree 0 on $BFV(E)$ encoding the pairing between $\mathcal{E}$ and $\mathcal{E}^*$. Denote the inclusion $\Gamma(\wedge E)\to BFV(E)$ by $\wedge p^*$, the projection $BFV(E)\to\Gamma(\wedge E)$ by $\wedge i^*$ and the homotopy introduced in the proof of Lemma 2.7 in Chapter 4 by h.*

Then

$$\wedge i^*\left([p^*(\xi_1),h([\cdots h([p^*(\xi_{(k-1)}),h([p^*(\xi_k),h(X)]_G)]_G)\cdots]_G)]_G\right)=\frac{1}{k!}(\xi_1\cdots\xi_k\cdot X)|_S$$

holds for arbitrary sections $\xi_1,\dots\xi_k$ of E and X and element of $BFV(E)$ of ghost-momentum degree 0, i.e. $\Gamma(\wedge\mathcal{E})$. Here $\xi_1\cdots\xi_k\cdot X$ is the fibre derivative of X interpreted as a vertical vector field on $E\to S$, see Remark 3.10 in Chapter 3.

Proof. Let $(x^\beta)_{\beta=1,\dots,s}$ be local coordinates on S, $(y^i)_{i=1,\dots,e}$ linear coordinates along the fibres of E, $(b^i)_{i=1,\dots,e}$ the corresponding local frame on $\mathcal{E}^*$ and $(c_i)_{i=1,\dots,e}$ the dual frame on $\mathcal{E}$.

We prove the Lemma inductively. For $k = 1$ it follows from

$$\begin{aligned}
\wedge i^*([\xi, h(X)]_G) &= \wedge i^*([\xi, \sum_{i=1}^{e} b^i \int_0^1 (\frac{\partial X}{\partial y^i}(x, t\cdot y, c))dt]_G) \\
&= \sum_{i=1}^{e} \wedge i^*(< \xi, b^i >) \int_0^1 (\frac{\partial X}{\partial y^i}(x, 0, c)dt) \\
&= \sum_{i=1}^{e} \wedge i^*(< \xi, b^i >) \frac{\partial X}{\partial y^i}(x, 0, c) \\
&= (\xi \cdot X)|_S.
\end{aligned}$$

Assume we established the claimed identity for $k < n$. First we compute

$$\begin{aligned}
&\wedge i^*([\xi_1, h([\xi_2, \cdots h([\xi_{(n-1)}, h([\xi_n, h(X)]_G)]_G)\cdots]_G)]_G) \\
&= \frac{1}{(n-1)!}(\xi_1 \cdots \xi_{(n-1)} \cdot ([\xi_n, h(X)]_G))|_S \\
&= \frac{1}{(n-1)!}(\xi_1 \cdots \xi_{(n-1)} \cdot \left(\sum_{i=1}^{e} < \xi, b^i > \int_0^1 \frac{\partial X}{\partial y^i}(x, t\cdot y, c)dt \right))|_S.
\end{aligned}$$

Observe that only the part of

$$\int_0^1 \frac{\partial X}{\partial y^i}(x, t\cdot y, c)dt$$

of polynomial degree $(n-1)$ with respect to the fibre coordinates $(y^i)_{i=1,\dots,e}$ contributes. Hence we can assume without loss of generality that $\frac{\partial X}{\partial y^i}$ is of polynomial degree $(n-1)$ with respect to the fibre coordinates. This implies

$$\begin{aligned}
&\frac{1}{(n-1)!}(\xi_1 \cdots \xi_{(n-1)} \cdot \left(\sum_{i=1}^{e} < \xi, b^i > \int_0^1 \frac{\partial X}{\partial y^i}(x, t\cdot y, c)dt \right))|_S \\
&= \frac{1}{(n-1)!}(\xi_1 \cdots \xi_{(n-1)} \cdot \left(\sum_{i=1}^{e} < \xi, b^i > \int_0^1 \frac{\partial X}{\partial y^i}(x, y, c)t^{(n-1)}dt \right))|_S \\
&= \frac{1}{(n-1)!}(\xi_1 \cdots \xi_{(n-1)} \cdot \left(\sum_{i=1}^{e} < \xi, b^i > \frac{\partial X}{\partial y^i}(x, y, c) \left(\int_0^1 t^{(n-1)}dt \right) \right))|_S \\
&= \frac{1}{n!}(\xi_1 \cdots \xi_{(n-1)} \cdot \xi_n \cdot X)|_S \\
&= \frac{1}{n!}(\xi_1 \cdots \xi_n \cdot X)|_S.
\end{aligned}$$

Consequently the claimed formula holds for all $n \geq 1$. □

LEMMA 12. *There is a smooth function* $\rho : [0,1] \to [0,1]$ *satisfying*

(i) $\rho(0) = 0$ *and* $\rho(1) = 1$,
(ii) ρ *is equal* $1/2$ *on* $[1/3, 2/3]$ *and*

(iii) *the restriction of* ρ *to* $]0,1/3[$ *and* $]2/3,1[$ *are diffeomorphisms to* $\rho(]0,1/3[)$ *and* $\rho(]1/3,1[)$ *respectively.*

Proof. First we construct a non-negative smooth function $\sigma:\mathbb{R}\to\mathbb{R}$ that vanishes on $[1/3,2/3]$. We define σ with the help of the smooth function

$$h(x):=\begin{cases}0 & x\leq 0\\ \exp\left(-\frac{1}{x}\right) & x>0\end{cases}$$

by setting $\sigma(x):=h(x-2/3)+h(1/3-x)$. On the complement of $[1/3,2/3]$ it is strictly positive. Finally we define

$$\rho(x):=\frac{\left(\int_0^x\sigma(\tau)d\tau\right)}{\left(\int_0^1\sigma(\tau)d\tau\right)}$$

for $x\in[0,1]$. The symmetry of $\sigma(x)$ with respect to $x=1/2$ implies that $\rho(1/2)=1/2$ holds. Moreover ρ has strictly positive derivative on the complement of $[1/3,2/3]$ and hence it is a local diffeomorphism there. Consequently its inverse on $]0,1/3[$ and on $]2/3,1[$ respectively is a smooth function. $\square$

Lemma 13. *Given two* n*-tuples* $0<a_1<\cdots<a_n<1$ *and* $0<b_1<\cdots<b_n<1$ *there is a smooth one-parameter family of diffeomorphisms* $(f_s)_{s\in[0,1]}$ *of* $[0,1]$ *relative to the boundary and starting at the identity such that* $f_1(a_i)=b_i$ *is satisfied for all* $i=1,\ldots,n$*,*

Proof. We prove a slightly more general result: let c be some real number in $]0,1[$ and $0<a_1<\cdots<a_n<c$, $0<b_1<\cdots b_n<c$ two tuples of points in $]0,c[$. We claim that there is a smooth one-parameter family of diffeomorphisms $(f_s)_{s\in[0,1]}$ of $[0,c]$ such that $f_1(a_i)=b_i$ holds for all $i=1,\ldots,n$ and that there is a open neighbourhood of $\{0\}\cup\{c\}$ where f_s is equal to id for all $s\in[0,1]$.

We first prove the claim for $n=1$, i.e. let a and b be two arbitrary elements of $]0,c[$. Choose a function κ that is 1 on the closed interval I with endpoints a and b and vanishes outside an open neighbourhoods of I in $]0,c[$. Define the vector field

$$X(s):=(b-a)\kappa(s)\frac{\partial}{\partial s}$$

It generates a smooth one-parameter family of diffeomorphisms $(f_s)_{s\in[0,1]}$ which starts at the identity and maps a to $a+s(b-a)$. In particular $f_1(a)=b$. Moreover there is a neighbourhood of $\{0\}\cup\{c\}$ where f_s is equal to id for all $s\in[0,1]$.

Now let $n>1$. First we find a smooth one-parameter family of $(g_s^n)_{s\in[0,1]}$ diffeomorphisms relative to the boundary and starting at the identity that maps a_n to b_n. Consider the $(n-1)$-tuples $0<b_1<\cdots<b_{(n-1)}<b_n$ and $0<g_1^n(a_1)<\cdots<g_1^n(a_{(n-1)})<b_n$ respectively. By induction hypothesis there is a smooth one-parameter family of diffeomorphisms $(h_s^{(n-1)})_{s\in[0,1]}$ of $[0,b_n]$ such that $h_1^{(n-1)}(g_1^n(a_i))=b_i$ for $i=1,\ldots,n-1$ and there is a open neighbourhood of

$\{0\} \cup \{b_n\}$ where $h_s^{(n-1)}$ is equal to id for all $s \in [0,1]$. We extend $h_s^{(n-1)}$ by the identity outside of $[0, b_n]$ and consider the smooth one-parameter family of diffeomorphisms $(f_s^n)_{s\in[0,1]}$ given by

$$\begin{cases} g^n_{\rho(s)} & 0 \leq s \leq 1/3 \\ g^n_1 & 1/3 \leq s \leq 2/3 \\ h^{(n-1)}_{2\rho(s)-1} \circ g^n_1 & 2/3 \leq s \leq 1 \end{cases}$$

where ρ is a gluing function, see Definition 3.1 in Chapter 5. By construction $(f_s^n)_{s\in[0,1]}$ starts at the identity, maps a_i to b_i for all $i = 1, \ldots, n$ and there is a open neighbourhood of $\{0\} \cup \{1\}$ where f_s^n is equal to id for all $s \in [0,1]$. $\square$

Bibliography

[AMR] R. Abraham, J. Marsden, T. Ratiu, *Manifolds, Tensor Analysis and Applications*, Applied Mathematical Sciences **75**, Springer (1988)

[BF] I.A. Batalin, E.S. Fradkin, *A generalized canonical formalism and quantization of reducible gauge theories*, Phys. Lett. **122B** (1983), 157–164

[BV] I.A. Batalin, G.S. Vilkovisky, *Relativistic S-matrix of dynamical systems with bosons and fermion constraints*, Phys. Lett. **69B** (1977), 309–312

[BRST] C. Becchi, A. Rouet, R. Stora, *Renormalization of the Abelian HiggsKibble Model*, Commun. Math. Phys. **42**, 127 (1975);
I. Tyutin, Lebedev Institute preprint N39 (1975).

[B] M. Bordemann, *The deformation quantization of certain super-Poisson brackets and BRST cohomology*, preprint, `math.QA/0003218`

[BGHHW] M. Bordemann, G. Ginot, G. Halbout, H.-C. Herbig, S. Waldmann, *Formalité G_∞ adaptée et star-représentations sur des sous-variétés coïsotropes*, preprint, `arXiv:math/0504276`

[CF] A. S. Cattaneo, G. Felder, *Relative formality theorem and quantisation of coisotropic submanifolds*, Adv. Math. **208** (2007), 521–548

[CF2] A. Cattaneo, G. Felder, *Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model*, Lett. Math. Phys. **69** (2004), 157–175

[CS] A. S. Cattaneo, F. Schätz, *Equivalences of higher derived brackets*, J. Pure Appl. Algebra **212** (2008), Issue 11, 2450–2460.

[dSW] A. da Silva, A. Weinstein, *Geometric models of noncommutative algebras*, Berkeley Mathematics Lecture Notes, AMS (1999)

[Fe] R. Fernandes, *Connections in Poisson Geometry I: Holonomy and Invariants*, J. Differential Geom. **54** (2000), 303–366

[Fl] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), no.3, 513–547

[FOOO] K. Fukaya, Y.-G. Oh, H. Ohta, , K. Ono, *Lagrangian intersection Floer theory - anomaly and obstruction*, preprint

[Gi] V. Ginzburg, *Momentum Mappings and Poisson Cohomology*, Int. J. Math., **7**, Nr. 3 (1986), 329–358

[Go] M. Gotay, *On coisotropic imbeddings of pre-symplectic manifolds*, Proc. Amer. Math. Soc., **84** (1982), 111–114

[Gr] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347

[GL] V.K.A.M. Gugenheim, L.A. Lambe, *Perturbation theory in differential homological algebra I*, Il. J. Math., **33** (1989)

[He] H.-C. Herbig, *Variations on homological Reduction*, Ph.D. Thesis (University of Frankfurt), `arXiv:0708.3598`

[Hi] M. Hirsch, *Differential Topology*, Graduate Texts in Mathematics **33**, Springer Verlag, New York (1994).

[KO] A. Kapustin, D. Orlov, *Remarks on A-branes, mirror symmetry and the Fukaya category*, J. Geom. Phys. **48** (2003), no. 1, 84–99

[KSo] M. Kontsevich, Y. Soibelman, *Homological mirror Symmetry and torus fibrations*, `math.SG/0011041`

[KSt] B. Kostant, S. Sternberg, *Symplectic Reduction, BRS Cohomology and Infinite Dimensional Clifford Algebras*, Annals of Physics, **176**, (1987), 49–113

[L] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbras de Lie associées.*, K. Differential Geometry, 12(2) (1997), 253–300

[MS] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Second Edition, Oxford Mathematical Monographs, Oxford University Press, New York (1998)

[Mc] K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Notes **213**, Cambridge Univ. Press (2005)

[Me] S. A. Merkulov, *Strongly homotopy algebras of a Kähler manifold*, Internat. Math. Res. Notices (1999), no.**3**, 153–164

[Mi] P. Michor, *Topics in Differential Geometry*, Graduate Studies in Mathematics, Vol. **93** American Mathematical Society, Providence (2008)

[N] N. Nakanishi, *Poisson cohomology of plane quadratic Poisson structures*, Publ. Res. Inst. Math. Sci., Volume **33**, Number 1 (1997), 73–89.

[OP] Y.G. Oh, J.S. Park, *Deformations of coisotropic submanifolds and strong homotopy Lie algebroids*, Invent. Math. **161** (2005), 287–360

[P] A. Pichereau, *Poisson (co)homology and isolated singularities*, J. Algebra **299** (2006), no. 2, 747–777

[R] M. Rothstein, *The structure of supersymplectic supermanifolds.*, in *Differential Geometric Methods in Mathematical Physics*, Lect. Notes Phys. **375** (1991), 331–343

[Roy1] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis (University of California in Berkeley) (1999), `math.DG/9910078`.

[Roy2] D. Roytenberg, *Poisson Cohomology of $SU(2)$-Covariant "Necklace" Poisson Structures on S^2*, J. Nonlinear Math. Phys. **9** (2002) (3), 347-356

[Sch1] F. Schätz, *BFV-complex and higher homotopy structures*, Commun. Math. Phys. **286** (2009), Issue 2, 399–443

[Sch2] F. Schätz, *Invariance of the BFV-complex*, preprint, `arXiv:0812.2357`

[Sch3] F. Schätz, *Moduli of coisotropic sections and the BFV-complex*, preprint, `arXiv:0903.4074`

[SS] M. Schlessinger, J. Stasheff, *The Lie algebra structure on tangent cohomology and deformation theory*, J. Pure Appl. Algebra **89** (1993), 231–235

[Sta1] J. Stasheff, *The intrinsic bracket on the deformation complex of an associative algebra*, J. Pure Appl. Algebra **89** (1993), 231–235.

[Sta2] J. Stasheff, *Homological reduction of constrained Poisson algebras*, J. Diff. Geom. **45** (1997), 221–240

[Ste] P. Stefan, *Accessible sets, orbits, and foliations with singularities*, Proc. London Math. Soc. **20** (1974), 699–713

[Su] H. J. Sussmann, *Orbit of families of vector fields and integrability of distributions*, Trans. Amer. Math. J. **180** (1973), 171–188

[V1] Th. Voronov, *Higher derived brackets and homotopy algebras*, J. Pure Appl. Algebra **202** (2005), Issues 1-3, 133–153.

[V2] Th. Voronov, *Higher derived brackets for arbitrary derivations*, Travaux Mathématiques **XVI** (2005), 163–186.

[W1] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. **6** (1971), 329–346.

[W2] A. Weinstien, *The local structure of Poisson manifolds*, J. Differential Geom. **12** (1983), 523–557

[W3] A. Weinstein, *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan **40** (1988), 705–727

[W4] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. **23** (1997), 379–394

[Z] M. Zambon, *Averaging techniques in Riemannian, symplectic and contact geometry*, Ph.D. Thesis (University of California at Berkeley)